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Developing & Describing the Use & Learning of Conceptual Models for Integer Addition and Subtraction of Grade 5 Students

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DEVELOPING AND DESCRIBING THE USE AND LEARNING

OF CONCEPTUAL MODELS FOR INTEGER

ADDITION AND SUBTRACTION

OF GRADE 5 STUDENTS

Nicole M. Wessman-Enzinger

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This dissertation reports the results of a teaching experiment, which explored student thinking about integer addition and subtraction. Through the lens of commognitive theory (Sfard, 2008), interpreting negative integers as secondary intuitions (Fischbein, 1987), and employing teaching experiment methodology (Steffe & Thompson, 2000), this study was a first step in developing more robust descriptions of students' conceptual models for integer addition and subtraction. I investigated: (a) the conceptual models that students exhibited, (b) the various ways that students utilized conceptual models while learning about the addition and subtraction of integers, and (c), the ways that students' conceptions evolved over the course of a teaching experiment. This study of students' conceptual models led to the modification and refinement of the CMIAS descriptions (Wessman-Enzinger & Mooney, 2014, 2015).

Three Grade 5 students were selected based on results of a pilot study and students' responses to a written assessment. Data for this study was used from the Open Number Sentence and Context Individual Sessions of the 12-week teaching experiment.

All of the Individual Sessions were videotaped and transcribed. The transcripts paired with all of the drawings produced by the students for stories for open number sentences generated and open number sentences solved during all Individual Sessions constituted the unit of analysis. A constant comparative method (Merriam, 1998) was used to modify the previous descriptions and develop new descriptions of the CMIAS (Wessman-Enzinger & Mooney, 2014). A description of the changes in mathematical discourse (i.e., word use, visual mediators, narratives, routines) was used to highlight the learning of integer addition and subtraction across the Individual Sessions.

There are seven CMIAS described in this study: Bookkeeping, Counterbalance, Translation, Relativity, Proceduralization, Analogy, and Algebraic Reasoning. These CMIAS were used differently in Individual Context Sessions and Individual Open Number Sentence Sessions. The three Grade 5 students also prominently utilized certain CMIAS over others. How the students used these CMIAS changed over time, these changes are considered to be descriptions of learning about integer addition and subtraction.

The results presented in this study extend the literature on student thinking about integer addition and subtraction by (a) describing student thinking within both contextual and symbolic problem types; (b) extending and modifying the previous descriptions of the CMIAS; and (c) providing a developmental perspective that includes learning over an extended period of time.

DEVELOPING AND DESCRIBING THE USE AND LEARNING OF CONCEPTUAL MODELS FOR INTEGER ADDITION AND SUBTRACTION OF GRADE 5 STUDENTS

NICOLE M. WESSMAN-ENZINGER

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I am thankful for my husband, Kyle Enzinger, for all of his love and support on this project and supporting me in all my pursuits. I am blessed to have him by my side through this entire journey. I know my dissertation is only better because of his constant love and support. I would also like to thank my parents, Warren and Denise Wessman, who taught me to love learning at a young age. If not for my parents, I wouldn't know about perseverance and dedication. Although our family has encountered many challenges during my time in graduate school, I am thankful for the opportunity to be able to accomplish this dissertation despite our circumstances. I have especially learned from my father that true strength is in how we decide to respond to our challenges.

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Although this dissertation is complete, there is still much more for me to learn and do about integers and myself.

"Now we see things imperfectly, like puzzling reflections in a mirror, but then we will see everything with perfect clarity. All that I know now is partial and incomplete, but then I will know everything completely, just as God now knows me completely."

1 Corinthians 13:12

N. M. E.

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THE PROBLEM AND ITS BACKGROUND

Learning about the negative integers is difficult because of their abstract nature. Although all numbers are considered abstract, learning about the negative integers demand a different realm of abstraction. Drake, an eighth-grade student I interviewed (Wessman-Enzinger & Mooney, 2014; Wessman-Enzinger & Mooney, 2015), reflected on his difficulties in abstracting negative integer concepts:

> You have the negatives like a thought thing. It's kind of mental. And, you can like literally take away so many apples or slices of pie from someone and you can still have it. And the other person would still end up having some. Whereas, negatives, if you have something and you take something away from them and they don't have any, you can still keep taking more. But, you don't really have anything. You still won't.

In this excerpt we see an eighth grader, with three years of experience operating with negative integers, struggling with the abstract nature of these numbers. Negative integers, as Drake pointed out, lack a way to be physically modeled with discrete objects in our world and are abstracted mathematical objects. The struggle students have in conceptualizing and operating with the negative integers is not a new or even surprising phenomenon to our field. For example, even Piaget (1948) wrote, "Everyone knows the difficulty that secondary students [and even university students!] have in understanding the algebraic rules of signs – 'minus times minus equals plus'" (p. 104). Historically, even famous mathematicians, such as Pascal and Diophantus, rejected the possibility of negative integer solutions to algebraic equations (Bishop, Lamb, Philipp, Schappelle, & Whitacre, 2011; Dehaene, 1997; Gallardo, 2002).

The negatives are a different kind of abstract because they are not naturally modeled by physical objects. The physical models that exist to make sense of the integers are limited (Peled & Carraher, 2008). That is, because many students use manipulatives as they learn whole numbers, it seems that an affordance of using physical objects with the teaching and learning of integers would be that students would be drawing upon something familiar. Yet, an obstacle of using physical objects with the integers is that the negatives integers have to be applied to the objects when the object physically exists. For example, one way of using physical objects to represent the integers to is to use a chip model where the negative integers are represented by red chips and positive integers by black chips (e.g., Liebeck, 1990). If one wants to physically model a negative integer, -*n*, then one has to consider the n objects that are physically present and countable as representing -*n* by extension that each countable object represents -1. Furthermore, another consequence of this type of modeling with physical objects is that some problems, such as 2 – -1 may not be an intuitive and modeling them with physical objects can be challenging (Vig, Murray, & Star, 2014).

Smith, Sera, and Gattuso (1988) maintained that knowledge "about relations, number, and natural categories are instances of knowledge that all human beings acquire from their daily interactions with the world and require to understand their world in the most basic ways" (p. 384). Through counting and experiences in the world, children develop conceptions of the natural numbers (e.g., 1, 2, 3, 4) and quantity (Piaget, 1952). Yet, students, like Drake, struggle to coordinate the negative integers with their environments (Wessman-Enzinger, 2013). In that sense, the negative integers are "un-natural" when compared to the natural numbers.

Context as a Pedagogical Tool

Teachers often use pedagogical objects, like Unfix cubes or Cuisenaire rods, when teaching students about number. However, pedagogical models for the teaching and learning of the integers are lacking (Peled & Carraher, 2008). The pedagogical models typically used with integers are the number line and chip model and each of these models for teaching have affordances and hindrances for the teaching and learning of integers (Vig, Murray, & Star, 2014). However, other pedagogical tools that are used in the teaching and learning of integers are typically number patterns and contexts. Perhaps number patterns and context are often utilized due to the lack of pedagogical models.

Because of the abstract nature of the integers and the lack of physical models, context is a pedagogical tool that is often employed to help students to learn and apply negative number concepts. Much of the research on the teaching and learning of integers, even with the chip and number line models, is often situated within various contexts. Battista (1983) used the context of electrons, represented by positive and negative chips, to teach negatives with "cancelling" properties (e.g., 2 + - 2 = 0). Stephan and Akyuz (2012) presented how the traditional context of assets and debits can be modified to net worth to help students discuss the negative integers. Research on the teaching and learning of integers is mostly situated in instructional experiences (e.g., Kilhamn, 2011; Liebeck, 1990), with different researchers using an array of contexts (e.g., balloons, motion, elevators). Even textbook authors from the 19th century into the 21st century have introduced the negative integers through contexts. For example, Brahmagupta, a mathematician and algebraist from India, utilized integers with the context of debts and fortunes in 620 AD (see, e.g., Liebeck, 1990). Brahmagupta used special signs for negative numbers and shared specific rules for positive and negative numbers contingent upon the context of debt and fortune:

> A debt minus zero is a debt.
>
> A fortune minus zero is a fortune.
>
> Zero minus zero is a zero.
>
> A debt subtracted from zero is a fortune.
>
> A fortune subtracted from zero is a debt.
>
> The product of zero multiplied by a debt or fortune is zero.
>
> The product of zero multiplied by zero is zero.
>
> The product or quotient of two fortunes is one fortune.
>
> The product or quotient of two debts is one fortune.
>
> The product of quotient of a debt and a fortune is a debt.
>
> The product or quotient of a fortune and a debt is a debt.
>
> (Brahmagupta, c. 629, as cited in NRICH Project, 2012)

In this excerpt, rules and operations about integers were presented in a context-specific manner. Extending these rules and uses of negative numbers to other contexts and to other mathematical ideas is not easy as evidenced by the historical struggle to make sense of and use integers (e.g., Gallardo, 2002; Heeffer, 2011). From a modern psychological perspective, how students make sense of integers, both within and outside of these contexts, and how students extend their use of integers to other advanced mathematical topics remain important questions for our field.

It is difficult to identify the role that these various contexts have on the teaching and learning of integers. The contexts used within the teaching and learning of integers are often criticized as being “contrived” (e.g., Peled & Carraher, 2008) and not capable of helping students learn how to extend their uses of integers to modeling realistic situations. For example, Ball (1993) described a “Magic-Peanuts model,” which she suggested created “hocus-pocus” thinking about integers. Yet, students struggle to use integers, even with other contexts that are “more realistic.” For example, Peled (1991) showed that students naturally used positive integers with elevation, rather than integers. In this study when students were asked to find the distance between a city that is 200 meters below sea level and a city that is 300 meters above sea level, most students calculated 200 + 300, rather than the conventional 300 – (-200). Similarly, Whitacre, Bishop, Lamp, Philipp, Schappelle, and Lewis (2012b) showed that seventh graders with experience with integers did not naturally use negative integers with a debit situation and only used positive integers.

Yet, researchers have shown that young children develop intuitions about negative integers from their every experiences (e.g., Hativa & Cohen, 1995). Additionally, Stephan and Akuyz (2012) have shown that attention to these contexts with modifications can help promote the use of negative integers. For example, they took the traditional credit/debit context with negative integers and brought attention to defining a person’s net worth. By highlighting the attention to net worth, students were able to use and learn about the negative integers successfully.

The issue of how negative integers emerge intuitively out of contexts, how contexts are useful for the teaching and learning of integers, and how integers are utilized in our world is complicated (Peled, Mukhopadhyay, Resnick, 1989; Mukhopadhyay, Resnick, & Schauble, 1990; Whitacre et al., 2012a, 2012b; Kilhamn, 2009, 2011). Understanding the role of context with integers is challenging, albeit important, but for mathematics teachers the use of context is a tool to help students learn how to operate with the integers and succeed mathematically. One of the challenges of using context with integers is that different contexts can promote different ways of thinking about integers.

Context as More Than a Pedagogical Tool

Contexts can illustrate important mathematical ideas and uses of the integers that are often lost in modern curriculum and standards (Wessman-Enzinger, 2015). Contexts used with integers promote ways of thinking that are more comprehensive than the context itself (Wessman-Enzinger & Mooney, 2014, 2015). Although the ideas about the role of context are complicated, it is important to look more broadly about what the thoughts or conceptions of contexts are promoting. Contexts, whether realistic or unrealistic, facilitate ways of thinking about the integers, or conceptual models of the integers. As the use of these contexts varies, how student may think about them may vary as well.

Consider the contexts offered for the teaching of integers in the *Common Core State Standards for Mathematics* (Council of Chief State School Officers [CCSSO] & National Governors Association [NGA], 2010). The CCSSO & NGA suggested uses of contexts include, "temperature above/below zero, elevation above/below sea level, credits/debits, [and] positive/negative electric charge" (p. 43). These contexts can promote important mathematical ideas about the integers. The use of both temperature above/below zero and elevation above/below sea level use the ideas of the *relativity* and *translation* of the integers (Wessman-Enzinger & Tobias, 2015). The creation of the temperature scale and the use of sea level represent relative measurements. With Fahrenheit and Celsius temperature scales, all temperatures are described relative to the zero degree. For example, the zero for both temperature and elevation represents an arbitrary zero, rather than an actual quantifiable zero. The zero degree for Fahrenheit is different from the zero degree for Celsius. The temperature, -2 degrees Fahrenheit is a description of temperature relative to the 0 degree Fahrenheit. Similarly, the use of sea level uses the idea of using the relativity of the integers. Temperature can also promote the use of *translation* when discussing temperatures dropping or rising. Similarly, the idea of elevation can also promote the use of translation when discussing airplanes rising in the air or dolphins jumping out of the water. Any context that involves moving and comparing those movements along a scale promotes idea of vectors or translations. The context of credits/debits promotes the idea of thinking of integers as increases or decreases in number, or *gains and losses*. Typically, the idea of credits and debits is used with money. Positive/negative electron charges promote the use of thinking about negative and positive integers as balancing each other out. The idea of positive/negative electrons is more than just "opposites." Positive and negative electron charges neutralize or *counterbalance* each other out. The use of the integers with electrons is also different from the other contexts. For example, if three electrons (-3) and three protons (+3) provide an electrical charge of 0, the mathematical equation is -3 + 3 = 0. The electrons, with a charge of -3, and the protons, with a charge of +3, still exist despite the neutralization. Whereas, with temperature -3 + 3 = 0 represents a temperature of three degrees below zero getting three degrees warmer. And, with credits/debits, -3 + 3=0 represents the situation if a debt of three is negated with a credit of three. The temperature or debt no longer exists, in contrast to thinking about cancelling in a counterbalance way where the quantities are still present and just neutralized. As provided by these examples, the use of context provides various mathematical ways to think about the use of integers in the equation -3 + 3 = 0 differently. The suggested contexts promote ideas of thinking about the negative integers as counterbalance, bookkeeping, translation, and relativity. In fact, these mathematical ways of thinking, as well as some others, emerged as Conceptual Models for Integers that students used to make sense of the integers (Wessman-Enzinger & Mooney, 2014, 2015). These conceptual models are described in detail next.

Discovering the Conceptual Models for Integer Addition and Subtraction

In my first research study about student thinking, I investigated how experienced learners connected contexts to integer addition and subtraction open number sentences (Wessman-Enzinger & Mooney, 2014, 2015). I initially worked with six Grade 8 students. Each of the students posed stories for ten integer addition and subtraction open number sentences. The stories that the students posed were examined for the overall mathematical thinking and use. Bell (1984) referenced that contexts may be isomorphic to larger ways of mathematically reasoning:

> It is also becoming clear (though I am not sure that this has been documented) that situations which are structurally identical when fully mathematized are by no

> means similar when first perceived; for example, money and temperature problems are differently perceived, although they both involve states and changes in directional quantities. It follows that we need to consider much more seriously than we have previously done, the development of conceptual structures in one context, then another, then perhaps exploring isomorphism. (p. 56)

The Conceptual Models for Integer Addition and Subtraction (CMIAS) emerged from this small study as a way to describe student thinking about the integers. I subsequently explored the stories that fifth graders posed (n = 131) and then pre-service teachers (n = 98). The CMIAS after their initial development from this work are described below.

Counterbalance

The conceptual model of *Counterbalance* is promoted if the negative and positive integers in the situation seem to balance each other out. This is similar to the "balanced metric" idea, dancing partners context, and chip modeling concepts in the literature (e.g., Battista, 1983; Dienes, 1964; Whitacre et al., 2012a). The Counterbalance conceptual model is conventionally used within the context of charges of electrons. The zero in the counterbalance conceptual model represents neutralization. Positive and negative numbers in the counterbalance conceptual model are not just opposites, but opposites that neutralize. What differentiates the Counterbalance conceptual model from other models, is that these quantities always remain in the Counterbalance conceptual model, even when neutralized. An example of the Counterbalance conceptual model can be seen in the following story posed by a student:

> Joe did some bad things in the past. He's trying to even out the bad things he has done in the past by doing good things. So far, he still has five bad things to re-pay

> for what he's done. And, he can think of 26 good deeds. He does the 26 and by the end of week he's evened out the bad deeds and did more than he expected. He did 21 more good deeds.

This story represents the number sentence, -5 + 26 = 21. The -5 represents bad things done in Joe's past and 26 represents the good deeds Joe has done. Although the bad deeds and good deeds "balance" each other out, they are still present. The deeds have been done either way and cannot be removed. The zero in this conceptual model represents the neutrality of having a balanced amount of good deeds and bad deeds.

Bookkeeping

The conceptual model of *Bookkeeping* is promoted if integers are used in a way to describe losses and gains. An example of a conventional use of the Bookkeeping conceptual model is the borrowing and gaining of money or credits and debits. Credit/debits of money are a prominent context in the teaching and learning of integers (e.g., Stephan & Akyuz, 2012). The zero in the Bookkeeping conceptual model represents having neither a specific gain nor loss. In this conceptual model the positive and negative integers simply represents a gain or loss of something and do not necessarily require the use of money as the context. For example, gains and losses can be conceptualized with "owing and gaining of candy bars" or "wanting and receiving of baseball cards." An example of the Bookkeeping conceptual model is illustrated in the following story:

> Lewis struggled when he needed to turn his homework in. He owed two assignments already. And later on in the day, he realized he still had three more assignment due. And in total he owed five assignments.

A student posed this story to represent the number sentence, -2 – 3 = 5. Here the debit or the loss is the assignments owed. Zero represents the status of not owing assignments.

Relativity

The conceptual model of *Relativity* is promoted if integers are used to describe relative, or arbitrary, positions. With the Relativity conceptual model, the zero represents the point of reference. For example, -2 could represent being two blocks away from home, where home represents zero. What distinguishes the Relativity conceptual model from other models is that the actual cardinality of the numbers, or quantities involved is not necessary. Consider the following story that demonstrates a Relativity conceptual model:

> Consider a baseball game. Suppose you are down five runs in the first inning and you end up losing by fifteen runs. You would have to have to be down ten runs in the other innings to be down by fifteen runs at the end of the game.

A student also presented this story to represent the open number sentence, -5 – □ = -15. In this story, we actually do not know how many runs that any particular team has earned. Rather, we only know what one team's score is relative to the other team's score. Here the negative integers represent being down runs and the positive integers represent being up runs relative to the other team. Zero in this model represents both teams with tied runs; and, this could be any amount of tied runs.

Translation

The conceptual model of *Translation* is promoted if the integers are treated as vectors or translations. With the Translation conceptual model, the integers used may shift any kind of mathematical object (e.g., a number, a point, a curve). The translation conceptual model often emerges from the contexts of travelling or moving about a linear model, coordinate plane, or three-dimensional space. The zero in this conceptual model is a zero vector or a translation of no movement. Similar to Relativity, the zero can represent any arbitrary point with the positive and negative number representing a translation in one direction or another from the relative zero. For example, when considering the temperature scale the 0, whether in Celsius or Fahrenheit, represents a once arbitrary choice, but not conventional use. When discussing temperature dropping or rising, this represents a translation on this scale. The following story, posed by a student, represents the number sentence, -14 + -7 = -21 and provides an example of the Translation conceptual model:

> You are going to your family's house for Christmas and you're travelling down the road. Let the negative numbers represent the miles that you travelled when you accidentally turn in the wrong direction. The further and further away you go in the wrong direction represents the larger the negative numbers. Suppose that you first you take a right and go negative fourteen miles away. And then, you take another right and go negative seven miles away. In total you went negative twenty-one miles away.

Here the negative integers represent moving in the wrong direction and zero represents being at the desired destination, or no movement away.

Rule

A conceptual model of *Rule* is promoted if negatives were used in a way that is contingent to some outside "rule" or algorithm. This rule may or may not exist outside the context of the problem or task. For example, students often notoriously apply an algorithm for subtracting a negative number called “keep-change-change.” The students who use this algorithm would treat the problem 2 – -3 as 2 + $^{+}$3. Students may also create rules that are not necessarily true. For example, some students reasoned as though negative integers can always be treated as if they were positive integers. The following story, posed by a student, for -2 – 3 = -5 provides an example of the rule conceptual model:

> There were negative two baseballs and three were subtracted. And there were negative five left. (-2 – 3 = □)

This was considered a Rule conceptual model because the student knew a procedure to obtain -2 – 3 = -5; however, the student was not able to apply a context that involved opposites to this number sentence because they were constrained by their use of Rule.

These are descriptions of the CMIAS that students used with the integers emerged from my previous study (Wessman-Enzinger & Mooney, 2014, 2015). This dissertation study suggests the need for the development and refinement of more robust descriptions of these conceptual models and for the investigation into other possible conceptual models. The previous data was limited in terms of developing the CMIAS because the data was only collected from stories that the students posed. Data did not provide insight into how students reason when solving problems situated in contexts common for integers or when solving open number sentences. Furthermore, how the CMIAS are connected and related needs to be unpacked. For example, the conceptual models of relativity and translation seem intimately related. Translation, which utilizes integers as movements, incorporates two uses of zero. One use of zero within the Translation conceptual model incorporates using zero as a translation of no movement and the other use of zero within the Translation conceptual model uses zero as a position, a relative position. Utilizing the Translation conceptual model may be contingent to utilizing zero as a relative position. More work about how these conceptual models evolve, how these models facilitate the learning about the operations with the integers, and how they may be related needs attention.

Use of Integers in Advanced Mathematics

In both current mathematics standards and research, it appears that the main goal learning about integers is to learn about how to operate with the integers (e.g., CCSSO & NGA, 2010; Whitacre et al., 2012a, 2012b). Operating with the integers is undoubtedly important for mathematics; however, students also need an understanding about negative numbers that transcends just operations about negative numbers. Understanding the mathematical uses of the integers that these conceptual models afford may help students in more advanced mathematical work. Some of the mathematical implications of these conceptual models of integers are discussed next.

Counterbalance

Negative numbers eventually are used to conceptualize negative areas between the x-axis and a curve that lies below the x-axis in calculus. When integrating in calculus or computing Riemann sums in analysis, there is a counterbalancing of positive and negative areas between curves and the x-axis. If the curve is above the x-axis the area is positive and if the curve is below the x-axis the area is negative. When the definite integral is 0 we know that the areas above the x-axis are equivalent to the areas below the x-axis. Although the result of an integral may be zero (see, e.g., Figure 1), the areas are still present, making use of the Counterbalance conceptual model.

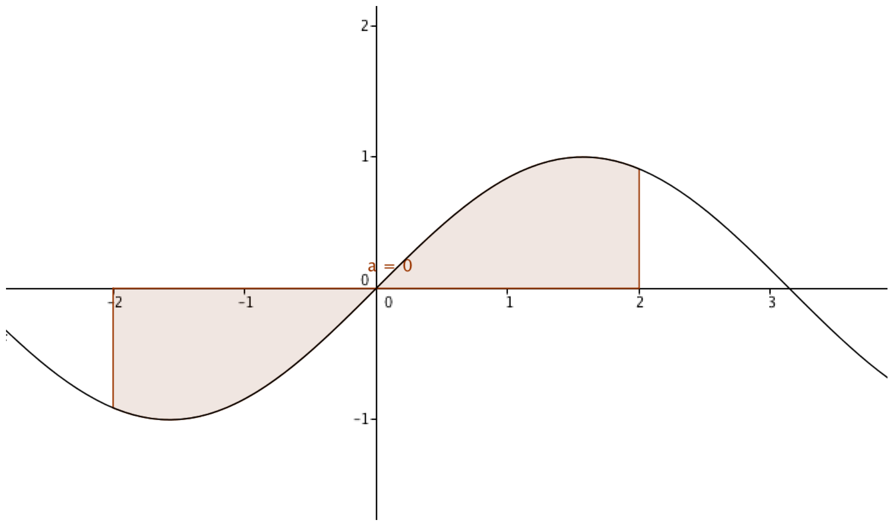


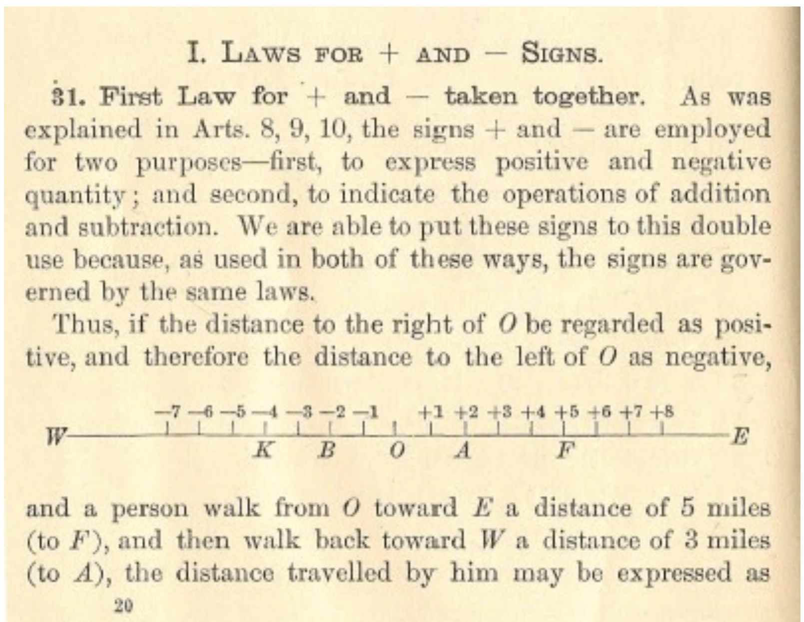
Figure 1. Example of Positive and Negative Areas Counterbalancing.

Bookkeeping

The ideas of afforded by the Bookkeeping conceptual model extend to mathematical uses beyond the traditional conventions of money. For example, consider statistics and computing means. If the mean of a data set is a certain amount and an element is added to that data set, then one might want to determine what that element needs to be for the mean to remain the same, increase by a certain amount, or decrease by a certain amount. This type of problem can be conceptualized and solved with a bookkeeping, or a gains and losses, perspective.

Relativity

Negative integers are found in algebra on the various axes as positions with Relativity. In a historical investigation of algebra texts written in the 1800s both the ideas of relativity and credits/debit of the negative integers were prominently discussed before operations with integers were introduced in the texts (Wessman-Enzinger, 2015). Before discussions about the rules of operating about the integers, authors of the nineteenth century texts discussed the nature of integers. For these authors, part of the nature of the integers was rooted in the relative nature of using the integers. For example, in my personal investigation one of the few number line illustration provided in these texts provided an arbitrary zero highlighting the relative positions of the integers (see Figure 2). Many of the nineteenth-century arithmetic and algebra authors maintained that the use of integers is relative.



I. LAWS FOR + AND − SIGNS.

31. First Law for + and − taken together. As was explained in Arts. 8, 9, 10, the signs + and − are employed for two purposes—first, to express positive and negative quantity; and second, to indicate the operations of addition and subtraction. We are able to put these signs to this double use because, as used in both of these ways, the signs are governed by the same laws.

Thus, if the distance to the right of *O* be regarded as positive, and therefore the distance to the left of *O* as negative,

−7 −6 −5 −4 −3 −2 −1 +1 +2 +3 +4 +5 +6 +7 +8

W *K* *B* *O* *A* *F* *E*

and a person walk from *O* toward *E* a distance of 5 miles (to *F*), and then walk back toward *W* a distance of 3 miles (to *A*), the distance travelled by him may be expressed as

20

Figure 2. A Relative Number Line (Day & Thomson, 1843, p. 20).

Although the contexts of temperature and elevation are utilized in our current mathematics curriculum (CCSSO & NGA, 2010; Whitacre et al., 2011), ideas like Relativity are absent and underdeveloped in the modern curriculum (Wessman-Enzinger, 2015). Yet, this idea of the relativity has implications in the learning of more advanced mathematical topics, like applying a relative position of the origin and Cartesian coordinate plane onto existing curves. Axes can be shifted and new planes, such as polar coordinates, can be introduced in order to obtain simpler equations for curves. The idea of axes and coordinate planes can be relative, just as the assignment of negative and positive numbers can be relative. For example, compare the graphs of $G(x) = x^2$ and $F(x) = (x-2)^2 + 3$. In Figure 3, $G(x)$ is blue and $F(x)$ is purple. Both the graphs of $G(x)$ and $F(x)$ have equivalent shapes. $F(x)$ is a translation of $G(x)$ two units to the right and three units up. However, instead of thinking of translating the graphs we think of creating a new coordinate system. Lines a and c, which are the green lines in Figure 3, could represent a new coordinate system and $F(x) = (x-2)^2 + 3$ could be redefined as $F(x') = (x')^2$ with this new coordinate system created with the new axes.

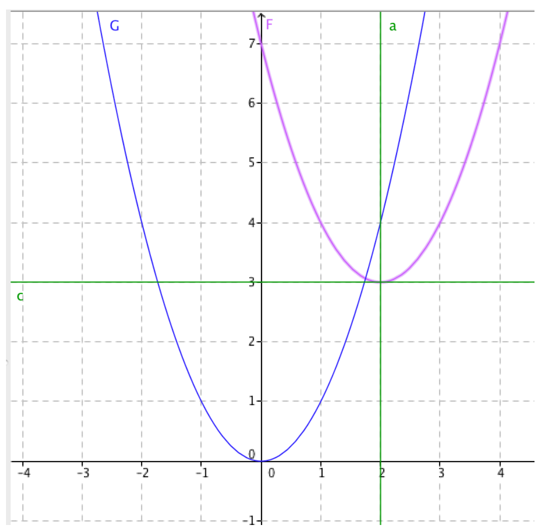


Figure 3. Example of Relativity with Coordinate Systems.

What axes are used or even what coordinate system (e.g., Cartesian, polar) are examples of Relativity in advanced mathematics.

Translation

Integers are used as transformations in algebra, geometry, trigonometry, and calculus when translating points, shapes, and curves. Eventually students learn that these transformations can be expressed by utilizing numbers as vectors or scalars, which they may use in physics and other science fields. Within this, the integers serve as both scalars and vectors. As students deal with geometric and algebraic transformations, they have to coordinate the use of integers both as scalars and as vectors. As students progress in more advanced mathematics they will eventually need to learn how to coordinate how using a negative integer as a scalar is different than using integers as a vector. Early experiences with directed numbers, or integers, and using Translation ways of thinking could help students as they progress mathematically.

Rules and Operations

After the introduction of integers in the sixth grade (CCSSO & NGA, 2010), the integers are a component of nearly every mathematical topic. Whether a student is solving an algebraic equation, working with matrices, graphing curves, or calculating statistics, a student will encounter the need for operating and working with integers efficiently. Students need conceptually developed rules and procedural fluency for operations with integers to be able to work with integers efficiently and effectively.

Exposure to different conceptual models (i.e., Counterbalance, Bookkeeping, Relativity, Translation, Rule) and developing rules for operations about integers has implications for students as they proceed in advanced mathematics. It is important for student to learn to operate efficiently about the integers, but students also need to conceptualize the various uses of the integers. For example, consider when students learn about fractions. Literature advocates for use of multiple models and flexibility between those models (e.g., area models, discrete models, circle models, fraction bars, number lines). Educators hope that students will use these models to learn about fractions at a deeper conceptual level and to extend their mathematical thinking. Thus, it follows that students need a variety of models, both physical and mental, for learning about all numbers, including integers. These different CMIAS not only describe the different ways of thinking about integers, but can also provide insight into learning what the various models of integers are. A deeper investigation of what these models are, how students develop them and how they are related needs to be carried out.

A Focus on Student Thinking

Despite the challenging nature of learning about negative numbers and their prevalent use in advanced mathematics, research focusing on student thinking about integers is often unconnected and is not as deeply investigated as the natural numbers. Because of this, there is an explicit need to link the students learning of natural number with negative integers (Bofferding, 2014). With the exception of emergent research agendas examining student thinking about integers and the addition and subtraction of negative integers (Bishop, Lamb, Philipp, Schappelle, & Whitacre, 2010, 2011; Bishop, Lamb, Philipp, Whitacre, Schappelle, & Lewis, 2014a; Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014b; Bofferding, 2010, 2011, 2012, 2014; Kilhamn, 2009, 2011), much of the research with negative integers is centered on teaching experiences (e.g., Saxe, Diakow, & Gearhart, 2013; Stephan & Akyuz, 2012; Vig, Murray, & Star, 2014). Most of the research about student thinking is centered on the ways that students think about addition and subtraction, with less emphasis with multiplication and division (e.g., Bishop et al., 2010, 2011, 2014a, 2014b; Bofferding, 2010, 2011, 2012, 2014; Gallardo, 1994, 1995, 2003), because it is important for our field to establish a well-researched, grounding within addition and subtraction. Since the research on student thinking about integers has been previously disconnected and emergent agendas are gathering footing, a consequence is noticing that many instructional experiences and research situated within these instructional experiences have not been based on student thinking because there are not many models of student thinking from which to base instructional and research decisions. Most of the relevant research has been situated within instructional experiences and is immersed whole-class instruction, which doesn't allow as much opportunity to investigate individual student thinking. Similarly, there is little research in student thinking of integers that documents learning over extended periods of time. This study extends the current research by focusing on the learning of addition and subtraction emerging from conceptual models and also by employing teaching experiment methodology to help bridge the student thinking and instructional gaps. Similarly, this study extends current research by providing a scholarly description of extended time with students.

With respect to the teaching of integers, various curriculum documents suggest various contexts to use and many curriculums use the same conventional contexts (Whitacre et al., 2011). However, research in the realm of student thinking has shown that students do not naturally use negative integers with these contexts (Mukhopadhyay, Resnick, & Schauble, 1990; Whitacre et al., 2012a, 2012b). Other researchers have pointed to other contexts being suitable for helping students grow conceptually with respect to negative integers in rich contexts (Stephan & Akyuz, 2012). Additionally, recent research conducted by this author has shown that students often make sense of negative integers in unconventional and unexpected ways (Wessman-Enzinger & Mooney, 2014). This dissertation study recognizes the complexity of the use of context with learning the integers and it directs attention to investigating the conceptions that students have and need about integers.

Problem Statement

Negative integers are used in our everyday world (e.g., debt, temperatures, golf, football yardage) and understanding integers is an important aspect of learning mathematics. Learning how to coordinate all of the mathematical uses of the integers (e.g., Bookkeeping, Counterbalance, Translation, Relativity) could be important for student success in advanced mathematics. However, many students focus strictly on the rules for operations with integers. For example, in a prior study students referenced a rule called "keep-change-change" for subtracting integers (Wessman-Enzinger & Mooney, 2014). This is not surprising since the focal point of most curricula and instruction about negative numbers is on operations. Although learning how to operate and construct rules for the operations is important, students need to understand the varied uses of integers, and we need to learn more about how they develop those understandings. This problem statement informed the development of my research agenda, which focuses on how students think about integers, the conceptions they have developed, and the conceptions they need to develop.

This dissertation is a first step in developing more robust descriptions of students' conceptual models for integer addition and subtraction. My research was guided by the following question:

> In what ways do fifth-grade students use conceptual models of the integers (e.g., Bookkeeping, Counterbalance, Translation, Relativity, Rule) as they (a) attempt to make sense of the integers and (b) learn about integer addition and subtraction?

More specifically, I investigated: (a) the conceptual models that students exhibited, (b) the various ways that students utilized conceptual models while learning about the addition and subtraction of integers, and (c), the ways that students' conceptions evolved over the course of a teaching experiment. This study of students' conceptual models led to the modification and refinement of the CMIAS descriptions (Wessman-Enzinger & Mooney, 2014, 2015).

Theoretical Perspectives

Intuitions

As Fischbein (1987) showed, some concepts do not seem to emerge naturally out of intuition. Fischbein distinguished between primary and secondary intuitions. A primary intuition is one that emerges from personal experiences. A secondary intuition is one that emerges from an instructional influence. The evidence that the learning of negative numbers is not a primary intuition that naturally emerges without the assistance of an instructional experience is found both from a historical perspective and from the literature.

The historical background of integers reflects negative integers as an abstract concept that did not develop historically as a primary intuitive perception. Mathematicians spent over 1500 years conceptualizing negative numbers as "real" numbers. Viewed from a historical perspective, it is not surprising that conceptions of negative integers cannot be expected to develop readily among students (Gallardo, 2002; Henley, 1999; Heeffer, 2011). Although adults may conventionally use negative numbers with debt of money, or with elevations below sea level, this is not a natural intuition for many children. For example, students may perceive debts as a positive numbers and not recognize that negative numbers can be used (Mukhopadhyay, Resnick, & Schauble, L., 1990; Whitacre et. al 2012b). Students often need instructional or educational promptings, such as a number line poster in a classroom or a task like 2 – 5 = □, to think about numbers differently, so that they can begin to use negative integers. Ekol (2010), after conducting a study where children used a dynamical geometry environment to operate with the integers, stated, “The outcome of operating with negative numbers is clearly not an intuitive one to elementary students because they cannot connect physical objects like they do with positive numbers” (p. 337). From what we know about the historical development and these studies, I contend that negative numbers emerge from a secondary intuition (Fishbein, 1987). Because I content that that negative integers are a secondary intuition, this study aimed to provide students with instructional experiences that explicitly used negative integers (e.g., open number sentences with negative integers) that promote thinking and learning about negative integers.

Commognitive Theory

Learning can be perceived as a change in mathematical discourse (Sfard, 2008). With commognitive theory, learning is defined as a “process of changing one’s discursive ways in a certain well-defined manner” (Sfard & Avigail, 2006, p. 4). In reference to negative integers, Sfard and Avigail (2006) stated, “a person who learns about negative numbers alters and extends her discursive skills as to become able to use this form of communication in solving mathematical problems” (p. 4). Although negative integers may be a different kind of abstraction and require an instructional experience for students (Fischbein, 1987), students do not create a new mathematical discourse or participate in an entirely new learning experience. Rather, they are likely to modify their mathematical discourse about whole numbers to accommodate the negative integers. This change in their mathematical discourse is evidence of their learning. Sfard considers discourse as a communication with oneself, thinking as communicating (Sfard, 2008) influenced by learning experiences. Thus, thinking mathematically is mathematical discourse. Because learning is defined as a change of discourse, this study aimed to provide extended time with students in order to investigate the change in mathematical thinking, or mathematical discourse.

An aim of this study is to provide descriptions of students' mathematical discourses around the learning of the addition and subtraction of integers. Sfard (2008) describes the need for this type of research, "It seems that in order to come to grips with these and similar phenomena, one needs to go beyond the Piagetian frame of mind" (p. 9). That is, Sfard points to the need to go beyond just classifying a student in a category and to consider the overall context in which the learning is immersed. Indeed, it is important to classify strategies that the students use with the integers in this study; however, a descriptive account of the mathematical discourse is also necessary. Sfard points to the main tenets of mathematical discourse: word use, visual mediators, narrative, and routines.

Word use. Sfard (2008) classifies a discourse as mathematical is if the discourse includes language that is mathematical. Within this study, students' mathematical word use about integers was examined, paired with other tenets of discourse, to describe their thinking. Examining word use will provide evidence of different uses of CMIAS or insight into things they may draw.

Visual mediators. Discourses are often focused about a medium, a concrete object, or artifact. As a part of mathematical discourse, visual mediators are produced (Sfard, 2008). Visual mediators with integers may be the mathematical symbols written by students or the drawings they produce to discuss their thinking or solve a mathematical problem. Sfard and Avigail (2006) state that these visual mediators are "part and parcel in the act of communication, and thus of the cognitive processes themselves" (p. 7). The drawings that the students produced in this study are considered in relationship to word use, as well. The visual mediators are a crucial component to examining the mathematical discourse and change in mathematical discourse, even if visual mediators are not part of the spoken discourse. For example, students drew empty number lines and tallies as ways to solve problems. Identifying and analyzing these types of visual mediators is an essential component to making sense of the mathematical discourse about integers.

Narratives. Sfard (2008) defines a narrative as, "a series of utterances, spoken or written, that is framed as a description of objects, or processes with or by objects and is subject to endorsement or rejection, that is, being labeled as 'true' or 'false'" (p. 300). She also defines utterances as a, "communicational act in language (this category includes written communicational acts long with the spoken ones)" (p. 302). Thus, the interpretation of narratives in this study is that the narratives will be uncovered by using the written text (i.e., visual mediators) produced by the students and the spoken words (i.e., word use). Sfard (2008) described students' narratives as including, but not limited to, mathematical definitions, theories, theorems, and properties formed as student interact with the integers. For this reason, narratives can be endorsed or rejected. That is, a student may develop a narrative that is rejected later. Because students are learning how to operate with integers, their narratives are being generated over time, changing, and will not be stated as "mathematical theories" as they are often done in "scholarly mathematical discourse" (Sfard, 2008, p. 134). Rather, the narratives in this study are the *mathematical uses* that the student employ, as evidenced by their utterances (i.e., word use, visual mediators). In this study, as students developed and made use of various CMIAS, these conceptual models served as a way of describing the students' mathematical uses or theory building, otherwise called narratives. The CMIAS are descriptions of conceptual thinking; and, the CMIAS represent of mathematical uses of integers. That is, the CMIAS describe mathematical conceptualizations, which simultaneously represents thinking *and* mathematical use. Commognitive theory presents tenets of discourse, like word use, as thinking. Thus, it then follows that mathematical uses, like the Translation and Rule of integers, are an important tenet of students' discourse, which is considered mathematical use or narratives, that also represents thinking.

Routines. Routines refer to the set of repetitive patterns in mathematical and nonmathematical activities. This includes the mathematical activity of the participants as they substantiate their mathematical narratives. Sfard (2008) points to the repetitive characteristics of discourse as routines. The idea is that some routines may be inherent and not explicitly communicated as an expectation. Another aspect of routines is identification of when and how the routines occur.

There may be specific routines that students typically draw upon, like a drawing they produce repetitively. Identifying and describing the discursive routines established by the students provides perspective into the overall mathematical discourse. A mathematical routine includes identifying certain patterns. For example, students may draw a number line routinely for solving certain open number sentences more than they do for solving other open number sentences. If students stop drawing particular visual mediators, then identifying their routine and changes in routines is important to describe their learning. These routines are important to analyze and make sense of alongside the other components of commognitive theory to make sense of the students' learning about integers.

Routines also establish repetition. Mathematical discourse is considered, "a collectively implemented activity that, when observed over time in its diverse manifestations, displays repetitiveness, and thus patterns" (Sfard, 2008, p. 195). Since learning is viewed as a change in this discourse, looking at where repetition breaks down can present moments of cognitive conflict. Identifying these and making sense of them points to important aspects of learning about the integers. For example, a break in a routine occurs if a student who typically uses Translation to add then struggles with subtraction. Examining the routines of in the learning of integers is important for this study.

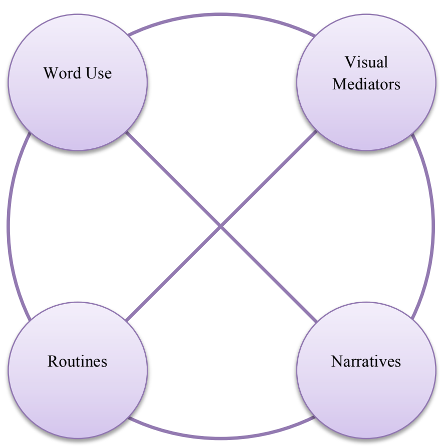


Figure 4. Relationship of the Tenets of Mathematical Discourse.

Word use, visual mediators, routines, and narratives comprise four components of discourse from a commognitive theory standpoint. Each of the components, although listed separately, relates to the other components (see Figure 4). For example, word use may work in conjunction with visual mediators, through a routine, to communicate a narrative. Although described separately, these tenets of commognition are synergistic and work together to describe students' discourses.

Significance of Study

Despite the profusion of context-specific research in mathematics education, research on student thinking about the addition and subtraction integers, and the emergent research agendas on student thinking about integers, research specifically positioned to address individual students' conceptions about integers in relationship to the learning over time is lacking at present. We need more insight into student thinking and learning about integers and what it truly means to learn about the integers over time. Because teaching and learning is dynamic and organic, evolving over time, this study will help inform both instruction and research. This study extends the body of knowledge of student thinking about integers by including a developmental perspective of student learning by investigating the conceptualizations students have about integer addition and subtraction for an extended amount of time with students. The significance of this study is that it extends the body of research with integers as an initiation to for research on negative numbers and rational number, in general.

This research extends the scholarly discussion about the role of context with integers by using the contexts to expose students to different mathematical ideas and examining what CMIAS students are using when solving open number sentences and applying contexts to open number sentences. This research focuses on the CMIAS emerging from students and studies how their CMIAS develop. In addition to learning more about how students think about negative numbers, this study presents an investigation into the different types of CMIASs by modifying this previous research with depth.

With any mathematical or conceptual model, models need to be used flexibly and interchangeably. The results from this study provide insight into flexibility, interconnectivity, and disconnections with various CMIAS about integers. A significant contribution of this study is the perspective for both researchers and teachers to look at the ways students use CMAIS differently when solving open number sentences and applying contexts to open number sentences.

Furthermore, although this study describes the thinking and learning of students, it is immersed in an instructional setting where CMIAS were used to develop tasks that used contexts to promote different mathematical ideas about integers. The CMIAS, which are ways of thinking about the integers, were not only used to describe student thinking in this study but they were also used as a pedagogical tool to determine which contexts to use to promote the teaching and learning of integers in this study. The use of the CMIAS as a pedagogical tool in this study can serve as an example for future design of other studies.

Organization of the Dissertation

This dissertation is organized into six chapters. This first chapter presented the problem statement and research question. Chapter II reports on the body of literature supporting the CMIAS and student thinking about the addition and subtraction of integers. Chapter III contains the methodology of this dissertation. This includes participant selection, pre- and post- assessments, structure of the individual and group sessions, and the development of the individual and group teaching episodes. Chapter III concludes with a detailed description of data analysis, which is structured around commognitive theory, which guided this study. The results of this study are reported in Chapter IV and Chapter V. Chapter IV presents the modified CMIAS from this study. Then, Chapter IV concludes with describing the CMIAS use of two participants, both when solving open number sentences and applying contexts to open number sentences. Results are reported in Chapter V and describe the CMAIS use for one participant, as well as, a descriptive snapshot of learning, or a change of discourse for that particular student. Chapter V concludes with a conjecture about the influence of the group session on the learning of that participant. The final chapter, Chapter VI, concludes with a discussion on the research findings. The discussion in Chapter VI also includes a synthesis of how this research connects to the other integer research agendas and their findings. Recommendations for both researchers and teachers are also provided in this concluding chapter.

CHAPTER II

REVIEW OF THE LITERATURE

This dissertation presents an investigation of student thinking about integer addition and subtraction to develop more robust descriptions of The Conceptual Models of Integer Addition and Subtraction ([CMIAS], Wessman-Enzinger & Mooney, 2014). This dissertation study also comes at an interesting time in the field of mathematics education where research about student thinking about negative numbers is of increased interest. Because of this, the first part of this chapter shares how I learned about the literature on student thinking about integers. It also shares my perspective of current research trends in student thinking about integers in the US. Then, the second part of this chapter provides a literature review to highlight the dichotomy of research on studying student thinking about integers: research situated in the realm of contexts and research situated in symbolism. Although an aim of this part of the chapter is to illustrate that the research on student thinking about integers and the addition and subtraction of integers in these two realms, this chapter also begins to connect the literature to the current definitions of the CMIAS. Addressing the research situated in symbolism, this chapter presents the seminal and influential work to this study and begins to connect it to CMIAS; addressing the research situated in contexts, this chapter is organized around initial descriptions that for CMIAS, which are investigated and further expounded upon in this dissertation study. An important feature of this entire chapter is that the literature is connected to each of the CMIAS. Understanding the literature in relationship to the initial definitions of the CMIAS provides deeper insight into the conceptual models and provides guidance as more robust descriptions of the CMAIS are modified and generated in this study.

My Personal Journey to Student Thinking about Negative Integers

When my own interests in student thinking about integers were emerging, I first examined issues of the *Journal for Research in Mathematics Education* (JRME) from the period 1970 to 2014 for research on student thinking about integers. According to Langrall (2013), “what is published in the journal is a reflection of what has been submitted. The continued success and growth of JRME is dependent upon the mathematics education research community in which it resides” (p. 4). If JRME is a reflection of the work in our field, then I reasoned that a thorough examination of the archives of articles published in JRME might give me a sense about the current research trends. I manually examined each article of JRME, volume by volume and page by page, published by JRME since its inauguration, noting whether it directly or indirectly incorporated the teaching and learning of integers, and in particular whether any attention was given to zero or negative integers.

Wheeler and Feghali (1983) presented a study about preservice teachers’ conceptions about zero, which represents the first and subsequently only study in JRME about conceptions of zero. Of course, beyond JRME, children’s conceptions of zero have been researched (e.g., Seidelmann, 2004) and have been shown connected to reasoning with negative integers (e.g., Bofferding & Alexander, 2011). But, continuing in JRME in sequential order, the first appearance of research with the negative integers appeared in

JRME with an investigation of two sixth grade students' reasoning about integer addition and subtraction in a microworld where a turtle made transformations on a number line (Thompson & Dreyfus, 1988). The second direct appearance of the negative integers in JRME was a proposed curricular model of instruction for addition and subtraction of negative integers using the context of net worth with assets and debts (Stephan & Akyuz, 2012). Subsequently, there were two articles about negative integers following Stephan and Akyuz, both published in 2014 (Bishop et al., 2014a; Bofferding, 2014). Bishop et al. (2014a) through a historical-epistemological theoretical lens provided Ways of Reasoning about open number sentences for integer addition and subtraction. Bofferding (2014) provided a perspective into the Mental Models that first-graders utilized with the negative integers, with a focus on ordering and comparing the integers, with addition and subtraction of integers also included. Considering Langrall's statement that JRME provides a perspective into the current state of the field and the large gap in time from the singular article in 1988 to the three current pieces in 2012 and 2014, this served as slight evidence to me, highlighting and supporting my feelings that there is increased interest in the mathematics education community about student thinking about integers.

Outside of JRME, most research over the past forty years with negative integers has been concentrated on instructional approaches (e.g., Battista, 1983, Linchevski & Williams, 1999). Although student thinking about negative integers has not been at the forefront of research in JRME until recently, there has been a constant interest in student thinking about integers throughout the decades (e.g., Marthe, 1979; Murray, 1985; Libeck, 1990; Gallardo, 2002). Although there is research over the past five decades about student thinking and integers, there are few of these research agendas investigating students' thinking about integers that have remained constant to build the field's knowledge up. Aurora Gallardo, a mathematics education researcher in Mexico, is one of the few researchers who has remained persistent in her research agenda on student thinking about integers. She has written papers on student thinking about integer spanning three decades (e.g., Gallardo & Rojano, 1987; Gallardo & Rojano, 1994; Gallardo, 2002; Gallardo & Hernandez, 2005; Gallardo, 2008). However, recently, significant research agendas on student thinking about integers have emerged, providing more descriptive insight into student thinking about integers by focusing on the role of context (e.g., Stephan & Akuyz, 2012), representations and instruction (Saxe, Diakow, & Gearhart, 2013), hindrances and affordances of models (Vig, Murray, & Star, 2014), the importance of understanding order and magnitude in relation to the integers and integer operations (Bofferding, 2014), and understanding the broad ways that students reason about open number sentences with integers (e.g., Bishop et al., 2014a).

Because of this increased interest in student thinking about integers in the field, the first working group on student thinking about negative integers was held at the 2013 Psychology of Mathematics Education in North America (PME-NA) meetings, which I attended. After participating in the discussion at this working group, Dr. Laura Bofferding and I manually conducted a collaborative review of both The Psychology of Mathematics Education (PME) and the Psychology of Mathematics Education in North America (PME-NA) proceedings from PME 1 to PME 38[1] and PME-NA 1 to PME-NA 36, spanning the years 1979 to 2014 in an effort to better understand what the field

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[1] A special thank you to Dr. Ken Clements, Dr. Nerida Ellerton, and the ISU Mathematics Library for access to many of these resources.

understands about student thinking about integers and discuss where it is potentially heading. In this investigation of the proceedings, one of the main findings was that there were 107 research reports that were directly or indirectly about integer research in both PME-NA and PME papers. The quantity and time span of these articles provided evidence that research into how students think about integers, and integer addition and subtraction, has been conducted for years, although not necessarily built upon. This literature review was presented at the second discussion group on integers, held at the joint 2014 PME & PME-NA meetings, and was part of a call about building bridges between integer agendas (Bofferding, Wessman-Enzinger, Gallardo, Salinas, & Peled, 2014).

The presentations and discussions by Bofferding and colleagues at this discussion group highlighted that research on student thinking began in our field by discussing typical struggles students had with integers (e.g., Guerrero & Martinez, 1982), identifying productive contexts, games, or models of integer instruction (e.g., Bell, O'Brien, & Shiu, 1980, Bell, 1982), and beginning efforts to identify problems types for integers (Marthe, 1979) and additive structures in general (Vergnaud, 1982). However, only some of these PME pieces have become seminal research pieces in the field (e.g., Javier, 1985; Peled, 1991). Javier (1985) and Peled (1991) are examples seminal PME pieces consistently cited in current research articles (e.g., Bishop et al., 2014a, 2014b; Stephan & Akuyz, 2012), an observation of typical citation practice in the integer field (Leatham, 2015). The differences between the proliferation of PME and PME-NA papers and the scarcity of articles situated about student thinking about integers in high-tier journals, like JRME, gave us a sense that there exists a disconnect in our research agendas on student thinking about integers, and was a topic of discussion at this working group. The disconnect between the agendas, the abundance of work on student thinking about integers present in the PME and PME-NA proceedings, and the apparent increased interest in student thinking about integers highlights a need for research agendas to attempt to connect their work on student thinking about integer addition and subtraction to others' work. Given my interest in developing the CMIAS, I worried that my own research agenda would also risk being disconnected from other's work. For this reason, I hoped to produce a dissertation study that illustrated an effort to connect this work to other relevant, well-established work.

The work of Bofferding (2014) serves as an example to me of research that has intentionally built upon the work of some of these prior studies on student thinking about integers, as she directly extended the work of some seminal integer pieces (Peled, Mukhopadhyay, & Resnick, 1989; Peled, 1991). Peled (1991) provides a taxonomy of thinking about the addition and subtraction of negative numbers and Peled et al. (1989) describes the mental models that students may draw upon when operating with the negative integers, specifically the divided number line and continuous number line. Bofferding (2012, 2014) extended this pioneering work by investigating first graders thinking about order and magnitude. Bofferding's work demonstrated that many first graders are capable of conceptions and developing mental models of negative integers and provided modified and new mental models (Bofferding, 2012, 2014), all the while also connecting and building off of the prior work of Peled et al. (1989). This current work by Bofferding provides an illustrative example of building and extending the prior work on student thinking about integers. Because this dissertation work is building and extending from my own previous investigations, this work served as an example that highlighted the need for my own work to be connected to this and others.

Because of this need for connection between agendas, this literature review hopes to explicitly highlight specific connections from other research agendas to the CMIAS. Specifically, this literature review connects to the agendas of both Bofferding (2014) and Bishop et al. (2014a) because they represent strong, emergent research on student thinking about integers. It is noteworthy to share that as I've progressed throughout this dissertation study, designing and proposing in 2013, this current research (Bofferding, 2014; Bishop et al., 2014a), which I am connecting to, simultaneously emerged. It is excellent timing to pair my own work to these current pieces to demonstrate an effort of building and connecting bridges between the research agendas that focus on student thinking about the addition and subtraction of integers, which is a need in the field (Bofferding et al., 2014).

Summarizing the CMIAS

Because the literature will be compared to the CMIAS throughout this chapter and one of the purposes of this dissertation study is to modify the CMIAS, brief descriptions of the CMIAS are described below.

Bookkeeping

The CMIAS *Bookkeeping* involves thinking about using integers to describe gains and losses. Zero in Bookkeeping represents a status of neither loss nor gain. Bookkeeping represents using the integers as a gain and loss of anything, and is not necessarily limited to the context of money. For example, gains and losses can be conceptualized with "owing and gaining of candy bars" or "wanting and receiving of baseball cards."

Counterbalance

The CMIAS *Counterbalance* involves thinking about and using the positive and negative integers in a way that balances or "cancels" each other out. This is similar to the "Balanced Metric" idea, cancellation, and chip modeling concepts in the literature (e.g., Battista 1983). The zero in Counterbalance indicates neutralization. The distinguishing element of Counterbalance is that the quantities always remain in the Counterbalance, even when neutralized. For example, consider three electrons (-3) and three protons (+3) that provide an electrical charge of 0, with -3 + 3 = 0. The electrons, with a charge of -3, and the protons, with a charge of +3, still *exist* despite the neutralization. Electrons and protons exist, but their charges are neutralized; chips still exist but their values are neutralized. This existence of the quantities that remain, but are neutralized, differentiates this way of thinking from Bookkeeping.

Relativity

The CMIAS *Relativity* involves thinking about and using integers in relative positions and as a comparison to a referent. With Relativity, the zero is not necessarily zero. The zero is a referent, or a point of reference or comparison. As seen in Molly's story, the actual score of the game is not known. Rather, the negative integers are used in a way relative to a tied game. For instance, Molly uses -5 to represent being down 5 runs from the winning score. The actual score could have been 25 and 20, 7 and 2, or any in which a particular team is down 5 runs.

Translation

The CMIAS *Translation* involves thinking about and using integers treated as vectors or directed numbers. With the translation conceptual model, the integers are used to shift any kind of mathematical object. The zero in this conceptual model may represent a zero vector, or no movement. Similar to Relativity, the zero can also represent a relative position with the positive and negative numbers representing a movement in one direction or another from the relative zero. Miguel demonstrated Translation in his story about travelling. For Miguel, the starting position is his house. Miles travelled in the direction towards a certain destination are positive and miles travelled in the opposite direction are negative.

Rule

The CMIAS *Rule* involves thinking about and using integers in a way that is contingent to some outside "rule" or algorithm and may be constructed outside the context of the problem or task. For example, students often apply an algorithm for subtracting a negative called "keep-change-change." This is a demonstration of Rule. Also, some students create their own "rules" that are not necessarily true for the minus sign.

The CMIAS were generated out of a research study that investigated student thinking where students posed stories, or applied a context, to various open number sentences. The rationale behind the particular study, where the CMIAS were born, came out of a need noticing the dualistically positioned research on student thinking about integers, where student thinking investigated in contextual situations (e.g., asset/debt situation) or symbolic situation (e.g., open number sentence).

Research on Student Thinking about Integers: Contextual & Symbolic

Research on student thinking about integer addition and subtraction supports what young children are capable of, productive reasoning about integers (e.g., Bofferding, 2014; Featherstone, 2000; Hativa & Cohen, 1995; Murray, 1985), and how they even invent their own notation for negative integers (e.g., Bishop, Lamb, Philipp, Schappelle, & Whitacre, 2011). Similarly recent research also illustrates specific ways of reasoning (WoR) that students may use to solve integer problems (Bishop et al., 2014a, 2014b). These aforementioned studies all investigated students' reasoning in symbolic settings. That is, they are asked to make sense of the integers themselves, add or subtract integers, or reason about integer addition and subtraction in open number sentences. However, there is also increased attention to the contexts that are utilized within the teaching and learning of negative integers (Linchevski & Williams, 1999; Mukhopadhyay, 1997; Stephan & Akyuz, 2012; Whitacre et al., 2012a; Whitacre et al., 2012b; Whitacre et al., 2014). Often addressing similar issues as the limited role of context, research concerning students' conceptions about negative integers and their metaphorical reasoning has also been approached (Chiu, 2001; Kilhamn, 2009; Kilhamn, 2011). Overall, the literature on student thinking about integers is often dichotomously separated into discussions about student thinking within contexts (i.e., situations connected to "real-life") or student thinking in symbolic settings (i.e., how students solve open number sentences or integer arithmetic sentences with integers). The research question of this dissertation study are located at the intersection of research on student thinking that emerges from contexts and research on student thinking that emerges from symbolic situation, because both constitute reasoning about integers.

Research on Student Thinking Emerging from Symbolic Representations

Thinking about and using the negative integers. The negative numbers, and the negative integers specifically for this study, are innocently deceptive. Although as mathematics educators or researchers we may utilize negative number daily, there are complexities to interpreting and understanding what the negative integers are. For example, both Gallardo (2002) and Bishop et al. (2014a) point to ways that we can think about and use negative integers.

Gallardo (2002) provided a framework for interpreting negative integers. She suggested that the negative integers can be interpreted in the following four ways as a subtrahend, a relative or directed number, an isolated number, or a formal negative numbers. Gallardo described subtrahend, as "the notion of number is subordinate to the magnitude (for example, in a – b, a is always greater than b where a and b are natural numbers" (p. 179). Gallardo's description includes interpreting the use of the negative integers as "subtraction." She described relative or directed numbers as, "the idea or opposite quantities in relation to a quality arises in the discrete domain and the idea of symmetry appears in the continuous domain" (p. 179). Typically, the integers, a set of discrete numbers, are plotted on the number line, a representation for a set of continuous numbers. However, she highlighted the symmetry of the integers in this continuous space. That is, the positive and negative integers are opposite and symmetric about zero. The negative integers may be conventionally to the left of zero on a horizontal number line. In this sense, the integers also have direction. The integers can represent a directed number, or vector, from 0 to -2. Gallardo also described an isolated number as "that or the results of an operation or as the solution to a problem or equation" (p. 179). Historically, experiencing the negative integers that emerge, as a solution to an equation, is where mathematicians were first confronted with the negatives. Although historically mathematicians may have struggled to accept the negatives and young children may have naïve conceptions about the negatives, solving a problem like 2 – 3 = □, presents an isolated number that has order that is "below" zero. Gallardo described the formal negative number, "a mathematical notion of negative number, within an enlarged concept of number embracing both positive and negative numbers (today's integers)" (p. 179).

Before discussing their study on student thinking about integer addition and subtraction, Bishop et al. (2014a) highlighted ways that -5 may be interpreted. Bishop et al. (2014a) stated that -5 may be interpreted as, "an action of removing 5 from a set" (p. 20). Because removing a number from a set, is similar to the take-away definition of subtraction, this interpretation can be compared to Gallardo's (2002) subtrahend definition. Bishop et al. (2014a) also stated that -5 can be interpreted as, "the integer between -6 and -4" (p. 20). In this description, -5 is a singular number with order. For this reason, the description of -5 as "the integer between -6 and -4" can be compared to Gallardo's (2002) isolated number definition. Bishop et al. (2014a) stated that -5 can be both interpreted as, "an action of moving 5 units left or five units down" and as "the location on a number line (coordinate plane, etc.) 5 units to the left of, or below, 0" (p. 20). These two descriptions are of the mathematical ideas of directed number and relative number, respectively. Although Gallardo (2002) has a comparable integer interpretation, "relative or directed number," the descriptions dually presented of location and movement distinguish the difference between relative and directed number. Bishop et al. (2014a) also stated that -5 can be interpreted as "the equivalence class [(0,5)] in which we define (a, b) to mean a – b, and all other ordered pairs (a,b) such that a + 5 = 0 include (1, 6), (2, 7), (100, 105), and all other ordered pairs (a,b) such at a + 5 = 0 + b for a,b that ∈ ℕ. [More formally, we can write (0, 5) ~ (a, b)" (p. 20). This is a contemporary mathematical interpretation of the integers and can be compared to Gallardo's (2002) interpretation of "formal negative number." Bishop et al. (2014a) concludes with interpreting -5 as "a representation of a $5 debt" (p. 20). Using -5 to represent $5 of debt may be interpreted either as a relative number or a direct number. For example, 5 can represent a debt or -5 can represent a debt because the use of number here is relative. Using -5 to represent a debt $5 is indicative of relative number use. However, debt can also be interpreted as a direct number, particularly if we consider it as a loss of money from 0.

Table 1 illustrates the definitions and comparisons of the negative integers provided by Gallardo (2002) and Bishop et al. (2014a). Imbedded in these ways of thinking about the integers are ways of using the integers that are more symbolic, while others are more contextual, and some lie on the bridge of symbolic and contextual. For example, interpreting -5 as an integer between -6 and -4 or as a part of a coordinate plane is inherently symbolic, while interpreting -5 as a debt is inherently contextual. Yet, reasoning about -5 as a debt in relationship to Gallardo's levels of interpretation of negative number becomes complicated. On one hand, it could be a directed number if one interprets "debt" as direction and the debit of $5 is the opposite of a credit of $5. On the other hand, one could interpret a debit of as a "loss" of five dollars. This loss can be represented symbolically by -5 or by 5. Since -5 is a relative number and -5 can represent a debt or loss of $5, but 5 can also represent a loss or debt of $5.

Table 1

Comparisons and Interpretations of Gallardo (2002) and Bishop et al. (2014a)

Gallardo (2002) interpretations of negative numbers (p. 179)	Bishop et al. (2014a) interpretations of -5 (p. 20)	Interpretations
Subtrahend “where the notion of number is subordinate to the magnitude (for example, in a – b, a is always greater than b where a and b are natural numbers”	“An action of removing 5 from a set”	Removing five from a set matches closely to interpreting a negative number as subtracting a positive number.
Relative or Directed Number “where the idea or opposite quantities in relation to a quality arises in the discrete domain and the idea of symmetry appears in the continuous domains”	“The location on a number line (coordinate plane, etc.) 5 units to the left of, or below, 0” “An action of moving 5 units left or five units down” “A debt of $5 is also a directed number; it is the opposite of a credit of $5.”	Placing a negative number on a number line allows one to interpret the negative number as a relative number or a directed number. Debt can be interpreted as direction. Or, -5 can be a relative number that represents a loss of five dollars.
Isolated Number “that of the results of an operation or as the solution to a problem or equation”	“The integer between -6 and -4”	The negative number may be treated as a symbolic number that has order.

Table Continues

Gallardo (2002) interpretations of negative numbers (p. 179)	Bishop et al. (2014a) interpretations of -5 (p. 20)	Interpretations
Formal Negative Number "a mathematical notion of negative number, within an enlarged concept of number embracing both positive and negative numbers (today's integers)"	"Describing the equivalence class [(0,5)] in which we define (a, b) to mean a – b, and all other ordered pairs (a,b) such that a + 5 = 0 include (1, 6), (2, 7), (100, 105), and all other ordered pairs (a,b) such at a + 5 = 0 + b for a,b that ∈ ℕ. [More formally, we can write (0, 5) ~ (a, b).]"	The negative number can be thought of in more formalized ways. For example, -5 is compared to an equivalence class. We can also talk about the additive group of the integers or the ring of the integers.

Relationship to CMIAS. When Bishop et al. (2014a) provided that -5 can be interpreted as a debit of $5, this represented a contextual interpretation of -5; however, there are other contextual interpretations as well. For example, -5 can represent an electrical charge, or -5 could represent a temperature. As discussed in Chapter I, and supported with discussion with the CMIAS, contextual interpretations have differences conceptually with the integers. Utilizing -5 as a temperature is treating the -5 as a relative number and utilizing -5 as an electrical charge is a quantity that incorporates neutralization (see, e.g., Counterbalance). Important mathematical ideas and uses of integers, such as this neutralization of quantities are lost when we interpret integers in isolation of operations.

The CCSSO & NGA (2010) recommendations for integer instruction parallel this in the sense that integers are introduced in the sixth grade without operations. After introduction of the integers, the CCSSO & NGA (2010) recommendations include the teaching and learning of all four operations with integers in the seventh grade, following integer introductions in the sixth grade. However, inherent in the interpretations of the negative integers both Gallardo (2002) and Bishop et al. (2014a) are ideas of operations. For example, in order to interpret an integer as a relative number, we need to understand ordering and numbers smaller than zero. The interpretation of the integers as a subtrahend or "an action of removing 5 from a set" points to subtraction, a mathematical operation. This points to an important question as a field: Is it possible to interpret or use the integers without operations? Even counting backwards by one with the integers is isomorphic to subtracting one. Integers may not be fully conceptualized without understanding and talking about operations, like addition and subtraction. Thus, the subsequent portions of this chapter focus on understanding thinking about integers through the lens of operations.

Thinking about the symbol (-). Symbolism is an important component to mathematical thinking and learning, and the symbol (-) may be considered an ambiguous symbol (e.g., Harkin & Rising, 1974; Gallardo & Rajano, 1994; Bofferding, 2014) with multiple meanings. The symbol (-) will henceforth be referred to as the "minus sign." Gallardo and Rojano (1994) and Bofferding (2014) provided three meanings of the minus sign: unary, binary, and symmetric. Table 2 illustrates these three definitions of the minus sign.

Vlassis (2008) presented obstacles that students had in the context of algebra with the multiple uses of the minus sign. Students often omitted the minus sign when solving algebraic equations. For example, the students may omit or ignore the minus sign with problems like $-2x = 6$ or $4 – x = 5$. Students also had difficulty with the minus sign when algebraic equations had positive solutions but incorporated a minus sign. For example, many students had errors when they solved problems like $12 - x = 7$. Vlassis also showed that the meanings of the minus sign shift as one solves algebraic equations. Bofferding (2010) demonstrated that these multiple meanings of the minus sign influenced the ways that children thought about addition and subtraction of integers. In a study with 22 Grade 2 students, where the children solved 17 addition and subtraction problems with integers, the children demonstrated these three uses (i.e., binary, unary, and symmetry) of the minus sign while solving addition and subtraction problems. Lamb et al. (2012) points that each of these meanings of the minus sign should be made explicit during instruction, whether operating with the integers or using algebra.

Table 2

Meanings of the Minus Sign

Role of the Minus Sign	Description
Binary	The minus sign is used to indicate subtraction between a minuend and subtrahend. This could be in arithmetic, where one finds the difference between two integers. Or, this could be in algebra where one establishes that subtracting a number is the same as adding it's opposite.
Unary	The minus sign establishes a formal negative number, relative or directed number, and isolated number or result/solution.
Symmetric	The minus sign indicates an opposite or an action to make opposite.

Relationship to CMIAS. The interpretation of the minus sign in relationship to the CMIAS is unclear and challenging. On one hand, it seems that using certain CMIAS may facilitate certain uses of the minus sign over others. For example, to think about the integers as Translation requires a unary interpretation of the minus sign in order to treat the integers as vectors. Similarly, the binary use of the minus sign seems related to Rule, as students often created their own ways for dealing with minus sign to operate with the integers, including interpreting the minus sign as subtraction. Yet, on the other hand, thinking about the integers productively with each of the CMIAS requires using the integers as opposites, which is a symmetric use of the minus symbol. Research into ways that understandings of the minus sign are related to the CMIAS or other ways of thinking about integer addition and subtraction are needed.

Thinking about order and magnitude. Research on student thinking about the ordering of the integers is less proliferated than the research on the addition and subtraction (e.g., Bofferding, 2014). Although Bofferding's work is representative of a recent investigation into student thinking about order and magnitude, it has evolved over time and is informed by past work.

Peled et al. (1989) conducted a study with children in first, third, fifth, seventh, and ninth grades, children in grade 3 to grade 9 were given a written assessment and interviewed on their knowledge of negative numbers. Some of the participants in the Peled et al. (1989) study were first graders, who were interviewed without a written assessment. None of the first graders in this study acknowledged the existence of negative numbers and none were able to perform operations with negative numbers. About half of the third and fifth graders were able to do operational problems with negative numbers; and, nearly all of the seventh and ninth graders were able to do the problems. The third and fifth graders often used "idiosyncratic rules," like -5 + 8 = -13.

The main results illustrated in this study are that children either utilize two different mental models: a continuous number line (CNL) or a divided number line (DNL). The CNL is similar to the standard real number line in mathematics. The DNL is a less coherent mathematical model. That is, students utilizing a DNL model treat the positive integers continuously and partition the number line at zero, struggling to do operations where one must “cross” zero. When operating with negative integers, the students referred to operating on the “negative side” or the “positive” side, not illustrated that one can operate with both sides.

Although minimal research was conducted mental models in mathematics education until recently (Bofferding, 2014), Varma and Schwartz (2011) provide an example of research from the cognitive sciences that supports investigations of mental models in symbolic settings by performing three experiments, with two of the three experiments conducted with adults and the third experiment conducted with children who have already received instruction on negative integers in school. In their study, Varma and Schwartz used comparison problems, given at varying magnitudes. Comparisons problems included integers that were both close together (e.g., -2 vs. 3) and far apart (e.g., -1 vs. 9). Both groups of adults were faster at the comparison problems that contained numbers that were near each other. The children were just as accurate as the adults, but did not exhibit as distinct of a difference in the time between the two different types of comparisons. Varma and Schwartz used this information to present another potential mental number line for integers. Since the children, who are freshly forming their mental number line in comparison to adults, did not have distinguishable answers in comparisons problems, Varma and Schwartz argue that this is evidence of a restructuring and development of a mental number line. The work of Varma and Schwartz supports work on comparison with the culturally advocated order prospective.

Although this work points to comparing integers with order, Bofferding (2010) discussed the conflict children may have with comparison problems when they are confounding order-based reasoning and magnitude-based reasoning. For example, consider comparing -8 and -2. The number -8 is more negative than -2 and -8 is “bigger” when using magnitude-based reasoning. Yet, -2 is closer to 0 on the number line than -8 is. In this case, -2 is considered “bigger” with order-based reasoning. Bofferding points to distinguishing between order-based reasoning[2] and magnitude-based reasoning.

Following the implication that both order and magnitude is important to understanding student thinking about operations (Bofferding, 2011), Bofferding reported on a student thinking about how first-graders reasoned about order (Bofferding, 2012, 2014). Bofferding (2014) highlighted important components of understanding order: counting backwards, filling in a number line, ordering inters and finding the greatest or least, deciding what is the greater integer.

Informed by Case’s (1996) theory of central conceptual structures for number (CCSN), Bofferding (2014) analyzed sixty-one first grader’s responses to order, value, and directed magnitude problems with integers and categorized their responses into a

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[2] Order-based reasoning is considered reasoning based on the sequential ordering of the integers (e.g., -3, -2, -1, 0, 1, 2, 3). Directed magnitude reasoning is considered reasoning based on movements and vectors, where -2 can be interpreted as a movement from 0 to -2. Magnitude-based reasoning is considered reasoning that is based on the absolute value of the numbers, without focus on direction.

series of mental models with extended robust descriptions of their reasoning. Bofferding described five different mental models and their subcategories for Integer Values and Order. These are described in three Mental Models: Initial Mental Models, Synthetic Mental Models, and Formal Mental Models, with two transition mental models (i.e., Transition I and Transition II), which are in between the Initial and Synthetic Mental Models and Synthetic and Formal Mental Models. There were two Initial Mental Models: Whole number and Absolute value. Students who exhibited used of the Initial: Whole Number Mental Model treated the negative integers as positive integers. Students who illustrated use of the Initial: Absolute Value Mental Model ordered negative integers, but treated them as positive values. For example, consider the integers -7, -5, 0, 3, 6 in this ascending order. Typical use of the Initial: Whole Number Mental Model includes ordering the integers 0, 3, -5, 6, -7. Whereas, typical use of the Initial: Absolute Value Mental Model includes ordering the integers, perhaps in a nonstandard manner, with a focus on value. For example, with this mental model a child may consider -8 to be greater than 1. Students, who provided evidence of the Transition I: Conflicted Value Mental Model, treated the integers as both negative and positive. For example, when comparing -5 and -1 a student would consider -5 > -1 because 5 is greater than 1 (positive number reasoning), but also state that both -5 and -1 are zero. Students that illustrated use of the Synthetic Mental Model used magnitude-based reasoning, while also recognizing that negative integers are less than zero. They recognized that 4 > -1, but also reasoned that -2 > -1. There were two different Transition II Mental Models: Dual Value and Unstable Integer. Students that provided evidence of the Transition II: Dual Value Mental Model were able to provide the correct value of the integers, but reversed the negative integers.

For the Transition II: Dual Value Mental Model, students used order-based reasoning for comparing numbers about half of the time. For Transition II: Unstable Mental Model, the students used the negative integers correctly, but were inconsistent a couple times. And, students classified with the Formal Integer Mental Model recognized the integers as symmetric around zero.

Bofferding (2014) also provided Mental Models for Directed Magnitude. These mental models were again categorized by Initial, Synthetic, and Formal. The Initial mental model for magnitude includes responses from students that were One-directional, where movement only occurred in one direction, or Direction-biased, where the focus was only on the direction and not the magnitude. Children's responses that were classified with the Synthetic model were magnitude-aware, where students were able to interpret movements as more or less in on direction or more or less in the opposite direction. The students' classified with the Formal mental model understood movements that were more or less in either direction.

Table 3 illustrates a comparison of the mental models described by Peled et al. (1989) and Bofferding (2014). Paired alongside their comparisons in Table 3 is a reflection on how Bofferding's mental models directly extend the work of Peled et al. (1989). The DNL described by Peled et al. (1989) is extended by Bofferding with descriptions of Initial, Transition, and Synthetic Mental Models. The descriptions of the Initial Mental Models distinguish between treating the whole numbers as positives and distinguishing the negatives as something different but ordering the negatives as positive numbers. Transition and Synthetic Mental Models distinguish a departure from Initial Mental Models, that's not quite DNL but not CNL either. The CNL from Peled et al.

(1989) and Bofferding's (2014) Formal: Integer Mental are similar. Bofferding extends the work of Peled et al. by describing some Transition II Mental Models that describe thinking that acknowledges that numbers are less than zero, but is not consistently developed yet. All of Bofferding's Directed Magnitude Mental Models extend the literature significantly on Mental Models by providing a set of mental models that go beyond describing just order-based thinking of children as she describes ways that children may perceive the magnitude of integers.

Table 3

Comparisons and Interpretations of Peled et al. (1989) and Bofferding (2014)

Peled et al. (1989) mental models	Bofferding (2014) mental models	Interpretations
Divided Number Line (DNL)	*Value and Order Mental Models* Initial: Whole number Initial: Absolute Value Transition I: Conflicted Value Synthetic: Magnitude	The DNL and Initial, Transition, and Synthetic models are related. The descriptions of the Initial Mental Models distinguish between treating the whole numbers as positives and distinguishing the negatives as something different but ordering the negatives as positive numbers. Transition and Synthetic Mental Models distinguish a departure from Initial Mental Models, that's not quite DNL but not CNL either.

Table Continues

Peled et al. (1989) mental models	Bofferding (2014) mental models	Interpretations
Continuous Number Line (CNL)	Transition II: Dual Value	The CNL and Formal: Integer Mental are similar.
	Transition II: Unstable Integer Formal: Integer	Bofferding extends the work of Peled et al. by describing some Transition II Mental Models that describe thinking that acknowledges that numbers are less than zero, but is not consistently developed yet.
	Directed Magnitude Mental Models Initial: One-directional Initial: Direction-biased Synthetic: Magnitude aware Formal: Directed magnitude	Bofferding extends the literature significantly on Mental Models by providing a set of mental models that goes beyond describing just order-based thinking of children as she describes ways that children may perceive the magnitude of integers.

Bofferding's (2014) Mental Models are consistent with others' recent work as well. For example, similar to the Initial: Absolute Value Mental Model, Bishop et al. (2011) described when the children first noticed a difference between positive integers and negative integers and separated them. Bishop et al. (2010) also described children who would classify -9 and 2 as different types of numbers, yet also consider -9 to be greater than 2. Chrysostomou and Mousoulides (2010) also found that even preservice teachers struggled with making sense of coordinating order and magnitude. The preservice teachers in their study debated about whether -500 is bigger than -700. When the preservice teachers compared -500 and -700 to debt, they would think that -700 was bigger; and, very few preservice teachers compared -500 and -700 to sea level or temperature contexts.

Relationship to CMIAS. The findings of Bofferding (2014) also inform the future development of the CMIAS (Wessman-Enzinger & Mooney, 2014) since both order and magnitude are a component of mathematical understanding within each of the CMIAS. Although all of the CMIAS incorporate order and magnitude, some of the CMIAS may support order-based more than magnitude-based reasoning. For example, ideas of order and directed magnitude are supported when counting forwards and backwards and this idea may readily extend to the Translation Conceptual Model, which is described as motion about a number line. However, the Relativity Conceptual Model, which is described as a comparison to other numbers, could possibly include the development or creation of a number line or number scale, which is contingent upon order. The Mental Models for Value and Order and Directed Magnitude (Bofferding, 2014) could be the mental models student draw upon as they begin to mathematically think about the integers with Translation and Relativity.

Other CMIAS support ideas of magnitude-based reasoning more, with less focus on the order and the directed magnitude. For example, magnitude-based reasoning seems to trump order-based reasoning within both Bookkeeping and Counterbalance. For example, Counterbalance represents a neutralizing space where all of the quantities remain present, and these quantities are often unordered and compared to each, with reasoning centered on the magnitude of the positive and negative quantities only. Using Counterbalance, we can think of -7 + 2 as seven negative ones (i.e., $7 \times (-1)$) and two positive ones (i.e., $2 \times (+1)$), comparing the seven negative ones to the two positive ones. Because there are five more negative ones after the two positive ones and two negative ones are compared and neutralized, we may reason that -7 + 2 = -5. In this case, magnitude-based reasoning worked well with order-based reasoning remained less prominent; however, culturally, when we compare two integers we value order-based reasoning.

In terms of comparing the Mental Models and CMIAS, the Mental Models could represent further descriptions of thinking within more magnitude-based reasoning (i.e., Bookkeeping, Counterbalance) and more order-based reasoning (i.e., Translation, Relativity). Although order-based reasoning is important culturally in school mathematics, it will be important in the future to connect what a well-established magnitude-based reasoning entails. Magnitude-based reasoning has important implications in advanced mathematics, particularly as students transition from learning the real-numbers to the complex numbers, where magnitude-based reasoning is of particular importance and relevance over order. Although the different Mental Models (i.e., Value and Order compared to Directed Magnitude) could represent important Mental Models for thinking with various CMIAS, both sets of Mental Models are important in all of the CMIAS, even if one is more prominent than the other. Value, order, and directed magnitude are important mathematical ideas inherent in all of the CMIAS. Figure 5 provides a visual image of how the Mental Models and CMIAS may be related.

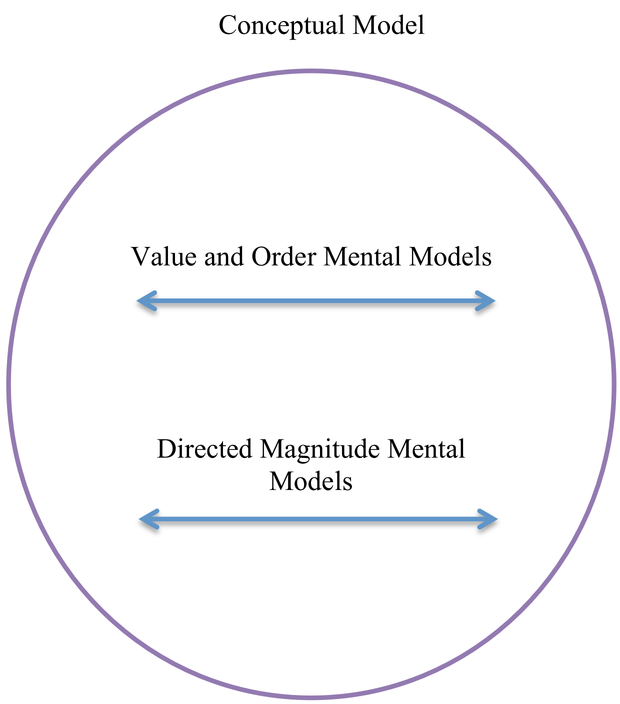


Figure 5. A Hypothesized Relationship of Mental Models and CMIAS.

Thinking about addition and subtraction. Research on student thinking about addition and subtraction has often focused on the correct or incorrect answers of students, illustrated problem types, highlighted challenges that students encounter with integer addition and subtraction, and illustrated some of the strategies that students use when solving integer arithmetic (e.g., Murray, 1985).

Challenges. Although many have advocated for young children's capabilities of working with negative integers (e.g., Davidson, 1987; Featherstone, 2000; Bofferding, 2014), much of the research on student thinking about integers is focused on later elementary grades to high school and university level. Davidson (1987) stated about learning the negative integers earlier:

> Introducing the integers in a manner which requires students periodically to overthrow their previously well-founded intuitions could contribute to this mystification of mathematics. Legitimizing non-positive numbers from the earliest level of instruction might facilitate constructing a more coherent and autonomous view of mathematics. (p. 431)

Perhaps the negative integers are challenging because when children learn about the negative integers in later grades, they have to often overcome over a decade of experiences of operating with positive numbers to accommodate the negative integers. Additionally, the negative integers are already naturally challenging due to their even more abstract nature comparative to the natural numbers, as pointed out in Chapter I and in the secondary intuitions section in Chapter II. Although the literature spans different ages and grade levels the commonalities of the challenges is consistent.

A challenge that students may encounter is that when students often make analogies from problems with only positive integers to problems with negative integers their analogies may break down. For example, students may incorrectly make an analogy, comparing 2 – 7 to 7 – 2 (Bofferding, 2010). Other students may think that problems like

2 – 7 are impossible (e.g., Bazzini, 1990; Booth, 1989). Similarly, students have to overcome challenges like subtraction is not commutative like addition. If students only use positive integers in instruction, subtraction should seem as if it is commutative if a student only uses positive integers (e.g., Bell, O'Brien, Shiu, 1980; Bofferding, 2011; Murray, 1985). Another challenge is that "more" and "less" have different meanings (e.g., Bell, 1984, Bofferding, 2010, 2014; Guerrero & Martinez, 1982). That is, as you move right along the number line numbers become "more" positive and "less" negative. Similarly, as you move left along the number line, numbers become "more" negative and "less" positive. And, another challenge that students have to coordinate is the use of the minus sign (i.e., "-"). As described earlier in this chapter, the minus sign has multiple meanings that students may confound. Similarly, students often operate with the negative integers by simply omitting or "ignoring" the minus sign and then adding it later (e.g., Ayres, 2000; Bell, O'Brien, Shiu, 1980). For example, to solve -5 – 4 a student may solve 5 – 4 = 1 and then apply a minus sign back in the problem. Also, signs may only denote locations, and not subtraction (e.g., Bell, 1984; Gallardo, 1995). For example, for problems like 10 – -2 students may interpret this as subtracting twice. That is, student may think that 10 – -2 = 10 – 2 – 2, rather than 10 – -2 representing the distance between two locations, 10 and -2 on a number line. Many researchers have highlighted the challenges that students have with changing directions and passing through zero on number line (e.g., Bell, O'Brien, Shiu, 1980; Bell, 1993; Bishop et al., 2014b; Bofferding, 2010, 2014; Bofferding & Hoffman, 2014; Gallardo, 1995). For example, problems like -1 – -3 may require students to think about crossing zero on the number line if the problem is interpreted as a movement from -1 three units to 2. However, if the

students interpreted -1 – -3 as the distance between -1 and -3 on the number line, it may not be as challenging for students. This points to another challenge where students have difficulty distinguishing between states and transformations with the integers (e.g., Marthe, 1982; Gallardo, 2003). That is, -2 can represent a state, like a temperature of -2 degrees Fahrenheit or a location on a number line, or -2 can represent a translation of dropping 2 degrees Fahrenheit or a movement two units lefts on a number line.

The challenges that students have when solving addition and subtraction problems with integers are summarized below:

- Analogies to whole numbers may break down, for example 2 – 7 may seem impossible to students (e.g., Bazzini, 1990; Booth, 1989)
- Subtraction is interpreted as commutative like addition (e.g., Bell, O'Brien, Shiu, 1980; Bofferding, 2011; Murray, 1985)
- More and less have different meanings with integers (e.g., Bell, 1984; Bofferding, 2010, 2014; Guerrero & Martinez, 1982)
- Operate with the negatives by omitting signs and then add the later (e.g., Ayres, 2000; Bell, O'Brien, Shiu, 1980)
- Signs may only denote locations, and not subtracting (e.g., Bell, 1984; Gallardo, 1995)
- Changing direction and passing through zero change the difficulty (e.g., Bell, O'Brien, Shiu, 1980; Bell, 1993; Bishop et al., 2014b; Bofferding, 2010, 2014; Bofferding & Hoffman, 2014; Gallardo, 1995)
- Difficulty distinguishing between states and transformations (e.g., Marthe, 1982; Gallardo, 2003)

Strategies. Although addition and subtraction with integers is challenging, not all subtraction problems are equally challenging (e.g., Hativa & Cohen, 1995; Marthe, 1982). Hativa and Cohen (1995) illustrated that some problem types are more intuitive than others. For example, problem types –a – -b, where a > b > 0, is an easier problem type than –a – b. Similarly, Marthe (1982) found that problems like x + b = c were more challenging for students when b and c had different signs. Overall, researchers have identified that different problem types for addition and subtraction of integers are of varying difficulty. However, how the problem types are considered varies. For example, Mukopdhyay, Resnick, & Schauble (1990) considered the following different problem types for addition and subtraction without magnitude (see Table 3), considering -2 + 5 the same problem type as -7 + 5. However, others (e.g., Peled, 1991; Bofferding, 2010) recognize that magnitude affects the problems type (see Table 4), differentiating between -2 +5 and -7 + 5.

Table 4

Problem Types for Integer Addition/Subtraction in Literature

Problem Types without Magnitude a, b > 0	Problem Types with Magnitude a, b, A, B, c > 0 where A > a and B > b
a + b	A + b, a + B, c + c
-a + -b	-A + -b, -a + -B, -c + -c
a + -b	A + -b, a + -B, c + -c
-a + b	-A + b, -a + B, -c + c

Table Continues

a – b	-A – b, -a – B, c – c
a – -b	A – -b, a – -B, c – -c
-a – b	-A – b, -a – B, -c – c
-a – -b	-A – -b, -a – -B, -c – -c

With magnitude considerations, there are 24 different problem types. The addition and subtraction problem listed in Table 4 do not include the different problem types for addition and subtraction open number sentences. For example, when open number sentences (e.g., -A – □ = c) are included, in addition to magnitude, the amount of different problem types increases substantially. The various open number sentences types for integer addition and subtraction will be discussed in Chapter III.

When considering the problem types in the second column of Table 4, Murray (1985) found that problem types like -A – -b, -a – -B, -A – b, and B – -a were the most challenging for students. Whereas, other researchers have found that the problem type -A – -b was not as challenging for students (e.g., Wheeler, Pearla, Bell, & Gattengo, 1981). Peled (1991) found that problem types A – -b and a – -B were the most challenging for students.

Within this large variety of different problem types for addition and subtraction with negative numbers, which have a varying level of difficulty, students have demonstrated use of a variety of strategies when solving these. One strategy that students use productively sometimes is the attaching or detaching of the minus sign to the answer at the end of the problem (e.g., Human & Murray, 1987; Vlassis, 2001). For example, a student my reason that -3 + 2 may be solved as 3 + 2 = 5 and then attach a minus sign at the end to obtain -5. Although this did not work, it may work for other problem types. For example, when solving -2 + -3 if a student uses a similar strategy they will end up with -5, which does work. Another strategy that students employ is interpreting the negative sign as subtraction (e.g., Gallardo, 1994; Human & Murray, 1987). For example, students may solve problems like 8 + -5 as 8 – 5 if they interpret the negative sign as subtraction. Although it can be a challenge that was described earlier, students often draw upon analogies to whole numbers productively to solve addition and subtraction problems with integers (e.g., Human & Murray, 1987; Murray, 1985). For example, students may compare -4 + -3 to 4 + 3 and reasoning that -4 + -3 = -7 because four of the same things plus three of the same things are seven of the same things. Similarly, students may use analogies to solve subtraction problems where the first number has a larger magnitude, like -4 – -3 = -1, by comparing it to 4 – 3 = 1. Students can also solve problems with negative numbers productively by drawing upon number line and movements (e.g., Guerroro & Martinez, 1982; Hativa & Cohen, 1995; Human & Murray, 1987; Murray, 1985; Poirier & Bednarz, 1991). For example, -5 + 2 can be solved by starting at -5 and moving two units to the right on the number line to -3. Also, students may utilize inverses to solve addition and subtraction problems (e.g., Human & Murray, 1987). For example, if student may recognize that 4 + -4 = 0. This may help the student solve problems like 7 + -4 because they may draw upon the use of inverse by decomposing the 7 to incorporate the inverse: 7 + -4 = (3 + 4) + -4 = 3 + (4 + -4) = 3. Students may also solve addition and subtraction problems with negative integers by comparing them to contexts, like temperature or debts and assets (e.g., Hativa & Cohen, 1995; Human & Murray, 1987). For example, students may compare -5 + 2 to a temperature context where it is negative five degrees out and the temperature increased two degrees. Students may also draw upon logically-based reasoning to solve addition and subtraction problems with integers (e.g., Bishop et al., 2014a; Murray, 1985). For example, a student may examine a string of subtraction problems like 4 – 3 = 1, 4 – 2 = 2, 4 – 1 = 4, and 4 – 0 = 4 and then logically conclude that 4 – -1 = 5 based on logic within in the pattern. When students encounter addition and subtraction problems with negative numbers, they may create rules to deal with the negatives, which may or may not work (e.g., Gallardo, 1994; Guerrero & Martinez, 1982; Ryan, Williams, Doig, 1998). As mentioned with the challenges, a student may create a rule that 10 – -2 is equivalent to subtracting twice or a student may memorize an algorithm m that 10 – -2 is equivalent to adding two. However, a student may invent a rule that in 10 + -2 the "plus sign" can be ignored.

The variety of strategies that students may employ when solving different integer addition and subtraction problems are summarized below:

- Attaching or detaching the minus sign to the answer at the end of the problem (e.g., Human & Murray, 1987; Vlassis, 2001)
- Interpreting the negative sign as subtraction (e.g., Gallardo, 1994; Human & Murray, 1987)
- Using analogies to whole numbers (e.g., Human & Murray, 1987; Murray, 1985)
- Drawing upon number line and movements (e.g., Guerroro & Martinez, 1982; Hativa & Cohen, 1995; Human & Murray, 1987; Murray, 1985; Poirier & Bednarz, 1991)

- Utilizing inverses (e.g., Human & Murray, 1987)
- Reasoning about arithmetic within contexts of temperature or debts and assets (e.g., Hativa & Cohen, 1995; Human & Murray, 1987)
- Drawing upon logically-based reasoning (e.g., Bishop et al., 2014a; Murray, 1985)
- Inventing rules which may or may not work (e.g., Gallardo, 1994; Guerrero & Martinez, 1982; Ryan, Williams, Doig, 1998)

Conceptions. Recent research on students conceptions and strategies for addition and subtraction of integers (Bofferding, 2010, 2011, 2013) and investigations into students' ways of reasoning when solving open number sentences with integers (Bishop et al., 2014a, 2014b) focuses on the productive ways that students can operate with the addition and subtraction of integers, but also point out ways that whole number reasoning can interfere with extending that knowledge to the integers. For example, curricula materials for mathematics in elementary school, including reform curricula, advocate for the use of "fact family" instruction. This type of instruction focuses on highlighting the relationships between operations that create a "family" of facts (e.g., 2+3 = 5, 5-3 =2). Bofferding (2011) challenged the present role of fact family instruction by advocating for the use of negative integers with young children. Addition is commutative over addition (e.g., 2+3=3+2=5); however, subtraction is not commutative over subtraction (e.g., 3–2 ≠ 2–3). Since negative numbers are not typically taught in first grade, many students developed the misconception that subtraction is commutative because addition is commutative. To address this issue, Bofferding conducted a teaching experiment with sixty-one first graders randomly assigned into three instructional groups. The first graders all took pre- and post- tests that were conducted as interviews. The instructional interventions were eight 45-minute lessons centered on concepts of negative numbers. Group 1 received instructional intervention that covered numerous topics: integer values, order, addition, and subtraction. Group 2 received an instructional intervention with only addition and subtraction. Group 3 also received an instructional intervention with two topics, integer value and order. Group1 and Group 3 made gains on the pre- and post-tests that were almost double the gains of Group 2, representing a major finding that emerged from this study. An implication from this study is that instruction on the order of negative numbers is possibly a critical component for development of the conception that subtraction is not commutative. This work points to a larger idea that students need more than strategies to solve addition and subtraction problems; rather, students need opportunity to develop productive mathematical conceptions about number through use of extending the whole number system to the integers.

Bishop et al. (2014a, 2014b) investigated students' conceptions of solving open number sentences with negative integers. Bishop et al. conducted clinical interviews with children from Kindergarten to high school, addressing both the novices and the supposed experts of negative integers. Many children, before formal instruction on negative integers, denied the existence of negative integers, similar to mathematicians of the past, classifying the problems as impossible. Other children categorized negative integers as a zero. For example, for problems like 3–4 and 2–7 the children would answer 0 in both cases. Some children would state that problems like $4 + \square = 3$ were "not real." Even middle school students have claimed that negative numbers are "not real." Bishop et al. (2014a) provide Ways of Reasoning (WoR) that children often used while solving these open number sentences. In the Bishop et al. study, clinical interviews were conducted for 50 to 70 minutes long with 47 children ranging from Grade 1 to Grade 4. The structure of these clinical interviews included: "introductory questions (asking children to name large/small numbers and to count backward), open number sentences, contextualized problems that could be solved using negative integers, and comparison problems" (p. 30). A major finding, consistent with Bofferding & Richardson (2013), is that children tended to use magnitude-based reasoning about the integers. That is, the children typically thought about the magnitude of -3 in comparison to the magnitude of 6 in the problem -3 + 6. In fact, Kilhamn (2009) argued that the ability to make numerical magnitude comparisons is an important component to understanding the integers. Although Bishop et al. (2014a) did provide contextualized problems to the students, their results reported on the ways that children reasoned while solving open number sentences with integer addition and subtraction. Bishop et al. (2014a) highlighted ordering relations, logical necessity and formalisms, magnitude, computation, and limited as the WoR that children used when solving open number sentences with integer addition and subtraction. The various WoR are described in the following section and compared in parallel to the CMIAS.

Relationship to CMIAS. Descriptions of each of the WoR and their relationship to the CMIAS is discussed next. In the WoR framework, Order is described as:

> In this way of reasoning, one leverages the sequential and ordered nature of numbers to reason about a problem. Using an order-based way of reasoning, one places integers in a sequence and can include the use of counting strategies or a

> number line with motion/movement. Counting strategies include counting forward or backward by ones (or another incrementing amount). (p. 32)

Order in the WoR framework seems similar to the description of Translation (Wessman-Enzinger & Mooney, 2014), in that it includes movement and motion about a number.

Magnitude, within the WoR framework, is described as:

> This way of reasoning is characterized by one's relating numbers and, in particular, negative numbers to a countable amount or quantity. Magnitude-based reasoning is tied to ideas about cardinality and the view of a number as having magnitude or substance. At times, negative numbers may be related to contexts (e.g., debt) or evoke the idea of opposite (directed) magnitudes. Opposite magnitudes include, for example, the ideas of (a) directional segments (e.g., vectors), (b) a time- certain event and the periods before and after this event has occurred, and (c) losing and gaining amounts. (Bishop et al., 2014a, p. 32)

Cardinality is emphasized in the Magnitude tenet of the WoR framework. Contextualized reasoning, such as debts and assets, is also included in this component of WoR. Additionally, a directed magnitude or vector is included in this WoR as well. This differs from the CMIAS for a couple reasons. First, reasoning about debts and assets would more than likely be part of the Bookkeeping or Counterbalance conceptual models (Wessman-Enzinger & Mooney, 2014), but depends on how the students reasoned about debts and assets. And, vectors and directed magnitudes are included in Translation. The Translation CMIAS groups together motion on a number line with vector reasoning; whereas, this reasoning is separated with WoR.

Logical necessity, within the WoR framework, is described as:

> In this way of reasoning, one takes a formal approach to problem solving, leveraging the ideas of structural similarity, well-defined expressions, and fundamental mathematical principles (e.g., commutativity, negation). This way of reasoning includes generalizing beyond a specific case by making a comparison to another, known, problem and appropriately adjusting one's heuristic so that the logic of the approach remains consistent. One may reason about a problem involving negative numbers (or make a generalization about operating with negative numbers) by making a comparison to a similar problem for which an answer is known and extending that reasoning to this new domain of negative numbers. (Bishop et al., 2014a, p. 32)

Students often make use of structural similarities when solving integer addition and subtraction problems (Bofferding, 2010). This way of thinking is highlighted in the WoR framework with Logical Necessity. Currently, ideas that point to Logical Necessity are not included in the initial descriptions of the CMIAS (Wessman-Enzinger & Mooney, 2014). Drawing upon this component of the literature to inform and strengthen the future development and modification of the CMIAS.

In the WoR framework, Computation is described as:

> In a computational way of reasoning, one uses a procedure, rule, or calculation to arrive at an answer to a problem involving negative numbers, either as part of the problem statement or as appearing in the solution set. Computational ways of reasoning about negative quantities can be present when solving a variety of algebraic and arithmetic problems, including solving systems of equations,

finding zeros of functions, and finding sums and products of negative values. (Bishop et al., 2014a, p. 32)

Computation relates to the reasoning that is described with the Rule Conceptual Model (Wessman-Enzinger & Mooney, 2014). Students' reasoning about integer addition and subtraction is often tied to procedures, rules, or calculations.

Limited, the last tenet of the WoR framework is described as:

> This category of reasoning reflects incomplete or limited views of negative numbers. At times, the domain of possible solutions is locally restricted to nonnegatives. Additionally, these strategies may not be based upon appropriate mathematical foundations. (p. 33)

Students that use this way reasoning may be tied so strictly to whole number reasoning that they are not able to reason about negative numbers productively. This component of WoR also does not directly relate to the current descriptions of the CMIAS. This is may be because the Conceptual Models were constructed based on the stories that fifth and eighth graders posed; whereas, the WoR framework was developed with data that included the solving of open number sentences. It may be that Limited is a way of thinking could be a part of Rule in the CMIAS. For example, if a student is reasoning about the negative integers as if they were positives, this could be based on their implicit rules that negative integers can be treated this way or that negative integers have the same rules as positive integers. Or, it may be that the Limited tenet of the WoR model can be layered or coupled with any of the other WoRs. That is, as Bofferding (2014) pointed out, some students may order negative integers as the following: -1, -2, -3, -4, 0, 1, 2, 3, 4. Students may use order-based reasoning, but this reasoning is "limited."

The descriptions of WoR (Bishop et al., 2014a) and the interpretations of WoR in relationship to the CMIAS are summarized below in Table 5. The discussion above highlighted that examining student thinking about negative integers is of increasing interest in the field. In fact, recent research agendas on student thinking about negative integers and the addition/subtraction of integers are emergent and rising (Bishop et al., 2010, 2011, 2014a, 2014b; Bofferding, 2010, 2011, 2012, 2014; Kilhamn, 2009, 2011). Discussion focused on student thinking in symbolic settings about: integers, order and value, directed magnitude, and addition and subtraction. Yet, much of the research with negative integers is centered on teaching experiences (e.g., Altiparmak & Özdoğan, 2010; Saxe, Diakow, & Gearhart, 2013; Stephan & Akyuz, 2012; Vig, Murray, & Star, 2014).

Table 5

Descriptions and Interpretations of WoR from Bishop et al. (2014a)

Ways of Reasoning (WoR)	Description and Interpretation
Order	Students use sequentially based reasoning, which includes counting strategies, motion, and movement. This seems directly related to the *Translation*.
Magnitude	Students use cardinality. This way of reasoning includes both contextual comparisons to debts and assets and directed magnitudes, or vectors. The use of directed magnitudes or vectors relates to the *Translation*. The use of contextual reasoning of debts and assets may be related to the *Bookkeeping* or the *Counterbalance*, depending on how the students utilize the quantities.

Table Continues

Logical necessity and formalisms	Students use structural similarities about problems and generalizations to solve the problems. This is not accounted for the in current descriptions of the CMIAS. This may be a component of the *Rule*, or it may help point to a different conceptual model.
Computation	Students use computations, rules, or procedures. This is seems directly related to *Rule*.
Limited	Students illustrate thinking incomplete or limited views of negative numbers. This may be part of the *Rule* or a different CMIAS.

When examining the literature on student thinking, most of the research focused on student thinking is centered on the ways that students think about addition and subtraction, with less emphasis with multiplication and division (e.g., Bishop et al., 2010, 2011, 2014a, 2014b; Bofferding, 2010, 2011, 2012, 2014; Gallardo, 1994, 1995, 2003), because it is important for our field to establish a well-researched, grounding within addition and subtraction. However, while the research on student thinking about addition and subtraction of integers is typically situated in investigating how the students make sense of and solve addition and subtraction problems (e.g., Bishop et al., 2014a, 2014b) and, in few unique cases, the ordering and comparing integers (Bofferding, 2014), there is less of a focus on student thinking situated in the realm of contexts and typical pedagogical models.

Research on Student Thinking Emerging from Contexts

Typical pedagogical models for the integers are limited to the chip model and the number line (e.g., Battista, 1983; Bolyard & Moyer-Packenham, 2006; Brasiel, 2011;

Saxe, Diakow, & Gearhart, 2013). Although these are the typical models for integer instruction, historically and currently we often use contexts a pedagogical tool to help students connect to the integers (e.g., Stephan & Akuyz, 2012).

The chip model represents a typical model for integers where different colored chips are used to represent demonstrate neutralization through quantities (e.g., Janvier, 1985; Liebeck 1990). For example, red chips could represent the negative integers and black chips could represent the positive integers. Thus, -2 +3 could be shown with three black chips and two red chips. The pair of one black and one red chip represents a zero pair and is “neutralized” (e.g., Vig, Murray, & Star, 2014). Similarly, using movements along the number line is another way to represent -2 +3 (e.g., Herbst, 1997; Wheeler, Nesher, Bell, & Gattegno, 1981). One could begin at -2 on the number line and then move three spaces to the right on the number line to 1. Besides the chip model and number line, other pedagogical models for teaching integers are limited in existence integers and are often used unsuccessfully (e.g., Peled & Carraher, 2008). Even when used successfully, these pedagogical models still have hindrances and affordances (Vig, Murray, & Star, 2014). For example, although both the number line and chip model can be used for problems like 3 – -2, the process of using them is more challenging and may not be intuitive for children.

Because these pedagogical models for integers have hindrances or breaking points, this may be one reason that educators turn to using contexts as a supplementary pedagogical tool for the integers. Although educators and researchers utilize contexts for understanding and facilitating student thinking and learning about integers, understanding the role of context within thinking and learning of the integers is complicated.

Researchers have productively utilized contexts to promote student thinking and learning with integer operations (e.g., Battista, 1983; Gallardo, 2002; Linchevski &Williams, 1999; Stephan & Akyuz, 2012; Whitacre et al., 2012a), contexts have often been criticized for being contrived (Ball, 1993) and not accessing authentic conceptions of integers. Researchers have also argued that children can take any context presented for negative integers and operate within the natural numbers instead (Whitacre et al., 2012b). In fact, this is reasonable claim, with the exception of temperature. Because a temperature scale with negative numbers is a cultural convention, negatives are required for use over natural numbers. For example, if we wonder what the new temperature is when the initial temperature is 2 degrees Fahrenheit or Celsius and drops 8 degrees a negative integer is a necessary consequence. If we have a fish below sea level 2 feet and a bird above sea level 8 feet and we are calculating the distance between the two, 8 + 2 is just as valid of a strategy as 8 – -2, but does not utilize negative integers. With the absence of a physical representation of negative integers existing in the world, children may intuitively draw upon their whole number experiences. In fact, reflecting on the negative numbers, mathematician Felix Klein (1925) stated, "for the first time, we meet the transition from concrete to formal mathematics. The complete mastery of this transition requires a higher-order ability in abstraction" (as cited in Varma & Schwartz, 2011, p. 363). Researchers, such as those with RME perspectives, recognize that contexts need to only be "real" to the students. In fact, Stephan and Akyuz (2012) provided an example where modifying the context of debts and assets to "net worth" provided a valuable instructional tool in helping students draw upon the negatives in the context. Whether contexts are contrived or not is trivial if the students accept the context and use of negative integers as authentic. For example, many students in Whitacre et al. (2012a) were able to use happy and sad thoughts productively with negative integers, which is not a typical context that is quantified in life. Whitacre et al. (2012b) asserted:

> Certainly, a mature understanding of integers includes the ability to relate them to quantities in the world. However, this does not entail that reasoning about real-world quantities like money should serve as a source for children's mathematical intuitions. On the contrary, we can imagine children developing a deep, purely mathematical understanding of integers. This understanding could then be superimposed upon real-world situations, in the way that we do as mathematically literate adults. (p. 963)

Understanding the role of context in relationship to the integers is complicated and has confounded our understandings of student thinking of integers in our field of mathematics education for decades (e.g., Bell, O'Brien, & Shui, 1980; Linchevski, & Williams, 1999; Mukhodpadhyay, 1997; Stephan & Akuyz, 2012; Whitacre et al., 2014). As a field, we need to dig in further into student thinking about integers in relationship to contexts. Over 30 years ago, Bell (1982) reflected on integers and context:

> Existing school course tend to begin by illustrating the concept of negative number in a number of contexts - temperature, co-ordinates, money, heights relative to sea level - then to introduce the number line, and to define the operation of addition with reference to the line only. Subtraction is occasionally defined as a displacement on the line, but more commonly as the addition of the additive inverse, or the "opposite." At least one course defines the first minus sign in 3 – (-4) as 'face west' and a second as 'walk backwards,' so that there is no

> operation sign at all. In subsequent work it is rare to find any discussion of the meaning of addition or subtraction of directed numbers in relation to any context, thought calculations such as (3 – -4)/(-5 – 1) are performed by rule, or perhaps by visualizing the displacements. There is a case for regarding the study of bank balances and transactions, relative heights, fast and slow clocks and combinations or additive and subtractive operators as worthy of a place in the curriculum, alongside the co-ordinate plane, because of the importance of these contexts, as well as for their value in providing for a fuller conceptualization of operations on directed numbers. (p. 208)

Bell's reflection from three decades ago still remains relevant to our field. Contexts are often used to draw students into the integers and learning the operations, with a focus on operation rules, remains at the zenith of curricular expectations. This is evidenced in the CCSSO & NGA recommendations. Integers are to be introduced without operations Grade 6. Although the students are not to learn about operations with integers in Grade 6, students are expected to connect the integers to contexts and make sense of them in these contexts. In Grade 7, recommendations include all four operations. As pointed out by Bell, students need opportunities to connect their reasoning with operations to context. The emergent literature on student thinking and context has begun to answer this by examining the mathematical models that have emerged from contexts (e.g., Stephan & Akuyz, 2012; Whitacre et al., 2012a, 2012b, 2014). Little attention has been spent examining the other direction, contexts that emerge out of mathematical models. In fact, the majority of research on the teaching and learning of negative integers situated within real-life contexts is conducted as teaching experiments. Little research exists that investigates individual student thinking and cognition of negative integers within these contexts (Kilpatrick, Swafford, & Findell, 2001; Whitacre et al., 2012a, 2012b). Yet, students' familiarity with context is critical if they are to reasonably connect the integers (Bell, O'Brien, Shiu, 1980). Research about student thinking about integers in context is dichotomously positioned between having students create contexts for number sentences and students using the integers in prescribed contexts where students use the integers.

Student-created contexts. Some examples of research that have addressed the contexts that students generate, rather than the mathematics that emerges from contexts, comes from Mukhopadhyay (1997), Kilhamn (2009), and Wessman-Enzinger & Mooney (2014). Mukhopadhyay (1997) used story-telling as a sense-making activity of negative numbers for children. Mukhopadhyay (1997) asked 32 students in grades 5, 6, and 7 to solve problems involving negative integers and tell a story that matched the equations. In this work, Mukhodopadhyay provided four case studies that demonstrated recognition that students struggled to generate stories. She hypothesized that this was attributed to the various mental models the students were possibly employing. Other researchers have investigated the stories that students pose for integer number sentences. Kilhamn (2009) also asked 99 preservice teachers to solve number sentences and describe their thinking for number sentences (e.g., -8–(-3)). Of the 99 prospective teachers, Kilhamn found that 71 used a rule to explain their reasoning, 5 had what was considered an irrelevant explanation, and 23 utilized metaphorical reasoning. Kilhamn viewed metaphorical reasoning as the use of a model or the utilization of a context to explain the mathematics. That means only 23 of the 99 students in Kilhman's study applied a contextual story to the mathematical models. Students who used metaphorical reasoning, or reasoning with contexts, either used number lines or temperature to explain their reasoning. This study was situated in Sweden, where the weather is often cold and temperature is measured in Celsius. For a significant portion of the year, the temperature in Sweden is in negative Celsius. It seems that this may be a reason why so many prospective teachers utilized the context of temperature in their explanations. Similarly, Roswell and Norwood (1999) also asked preservice teachers to pose stories for number sentences involving negative integers. All of the preservice teachers in their study posed stories about money/debt, temperature, height, or they used no context at all. The preservice teachers often changed the number sentence itself before posing the story.

Wessman-Enzinger and Mooney (2014) extended the work on student thinking in contexts (e.g., Whitacre et al., 2014) by focusing on how students attach contexts to number sentences and connecting it to the work of Chui (2001), Mukhodpadhyay (1997), Kilhman (2009), and Roswell and Norwood (1999) by investigating the conceptual underpinnings that could be attached to reasoning about integers by looking the stories that student posed for different number sentences. In fact, in this study the contexts that students used with addition and subtraction of negative integers were examined as an impetus to investigate the ways that students connect mathematical models to contexts, by asking the students to pose stories for open number sentences involving negative integers. One of the early researchers on student thinking about integers, Bell (1984), wrote:

> It is also becoming clear (though I am not sure that this has been documented) that situations which are structurally identical when fully mathematized are by no means similar when first perceived; for example, money and temperature

> problems are differently perceived, although they both involve states and changes in directional quantities. It follows that we need to consider much more seriously than we have previously done, the development of conceptual structures in one context, then another, then perhaps exploring isomorphism. (p. 56).

In this prior work, we desired to investigate the potential isomorphism of contexts and conceptual reasoning, which are connected to didactical and mathematical models through problem posing. Bell, O'Brien, and Shiu (1980) reflected that contexts utilizing integers, "problems in contexts differing like money and journeys appear to be handled in substantially different ways" by children (p. 122). For this reason, Bell (1993) designed a study where different contexts were represented in the different parts of the experimental design. Shore (2005) also provided an example that implicitly points to these potential isomorphisms. Shore provided two different contexts for a lesson on integer addition and subtraction, a voting context and good and bad deeds, which are connected to the chip model in instruction. In fact, Shore titled a section of her paper, "One Model, Many Contexts," highlighting that using contexts with integer addition and subtraction may support other models or other ideas about integer addition and subtraction. Even though research on metaphorical reasoning (e.g., Chui, 2001) unpacks how the students reason within contexts and there is an increased interest in researching student thinking about integers (e.g., Bishop et al., 2014; Bofferding, 2014; Whitacre et al., 2014), the CMIAS (Wessman-Enzinger & Mooney, 2014), which are being investigated in this study, also address the isomorphisms of contexts and reasoning that that Bell (1984) challenged our field to identify.

It is important to consider the "mathematics of children" versus the "mathematics for children" (Steffe & Olive, 2010). As a component of a large research project, Steffe and Olive discuss that the mathematics of children is distinctly unique from the mathematics knowledge possessed for the children. Steffe and Olive, do not address "mental models," but rather discuss that essentially some mathematics, or the "mathematics of children," is inaccessible, referring to mental models as first-order models that we will never be able to uncover. Rather the mental models or conceptual models that are developed are always "second-order" interpretations of the mathematics that the children bring to the table. Conversely, "mathematics for children" is the mathematics that is deemed appropriate by adults for instruction. In this dissertation study, the negative integers and learning to operate with them and use contexts with them are deemed the mathematics for children and the conceptual models of integers represent these second-order interpretations of their thinking. The following portion of the literature will identify some of the prominent literature that contributes and informs each of the CMIAS, specifically focusing on research that utilizes prescribed contexts and looks at the thinking and learning of integers from contexts.

Research with prescribed contexts informing Bookkeeping. Bookkeeping was the most utilized by students that posed stories for integer addition and subtraction open number sentences (Wessman-Enzinger & Mooney, 2014). This may be because Bookkeeping is supported by contexts that are easiest for students to deal with (Bruno & Martinon, 1996). The bookkeeping conceptual model is utilized when students use the integers as a gain or loss of anything, and is not limited to money. Students often use the bookkeeping conceptual model with unconventional contexts, such as wanting and obtaining baseball cards or losing and finding pencils (Wessman-Enzinger & Mooney, 2014). The Bookkeeping CMIAS, although not limited to thinking about the integers as gains and loses with money, is often utilized with money in many modern mathematics books (Whitacre et al., 2011), as well as, many historical texts (Hefendehl-Hebeker, 1991). In fact, most of the literature supporting evidence of Bookkeeping utilizes money (e.g., Liebeck & Williams, 1990; Whitacre et al., 2012b, 2014).

Research into the ways that students reason within Bookkeeping shows that students struggle with applying the negative numbers to those contexts (Whitacre et al., 2012b, 2014) and these contexts may promote incorrect conceptions about integers (Mukhopadhyay, Resnick, Schauble, 1990). Whitacre et al. (2012b, 2014) provide an example of current research with a bookkeeping perspective. Forty students in grade 7 were interviewed and their intuitions involving a context of money analyzed (Whitacre, 2012b). Students were given the problem, "Yesterday, you borrowed $8 from a friend to buy a school t-shirt. Today, you borrowed another $5 from the same friend to buy lunch. What's the situation now?" (Whitacre et al., 2012b, p. 959). Students responded with three different ways of reasoning: conventional, unconventional, and perspectiveless. Conventional reasoning is considered to be reasoning that is aligned with what is typically found in textbooks. That is, the number sentence is seen from the borrower's perspective of *owing* money and written -8 + -5 = -13. Unconventional reasoning, not aligning to reasoning in textbooks, is considered to be money that is *gained* by the borrower and is written 8 + 5 = 13. Perspectiveless reasoning is 8 + 5 = 13, where 8 represents $8, 5 represents $5, and the number 13 represents $13. Although the use of the word perspectiveless could be criticized since students who write this number sentence definitely have some sort of perspective, the authors chose this word for the type of reasoning because the child is reasoning in a way in which the perspective of borrower or lender is not informing the way that they have chosen to solve the problem. From this perspectiveless viewpoint, the representation of $8 + 5 = 13$ is used to communicate both what is owed and what is borrowed. This number sentence can be representative of both the borrower and the lender. All of the forty 7th graders solved the problem correctly; however, 33 or 82.5% of the seventh graders wrote $8 + 5 = 13$ as one of their equations and usually the first one too. Because of this, Whitacre et al. (2012b) argued that student intuitions about negative numbers do not necessarily evolve from contexts. The implications of these findings include a re-evaluation of the role of context in instruction and that instruction does not need to necessarily begin with "real-world" contexts. Whitacre et al. (2012b) reflected, "On the contrary, we can imagine children developing a deep, purely mathematical understanding of integers. This understanding could then be superimposed upon real-world situations, in the way that we do as mathematically literate adults" (p. 963). This type of Bookkeeping discussion can be related to Relativity. That is, because integers are relative numbers students may write number sentences either for positive or negative integers for contextual problems like, "Yesterday, you borrowed \$8 from a friend to buy a school t-shirt. Today, you borrowed another \$5 from the same friend to buy lunch. What's the situation now?" That is, $8 – 5$ is just as appropriate as $-8 + 5$ to model the situation, although $8 – 5 = 3$ is not mathematically equivalent to $-8 + 5 = -3$. Students who write a number sentence for negative integers are no more correct or mathematically advanced than students who write a number sentence with positive integers. Rather, because integers are relative numbers, both number sentences are correct. And, recognizing that both number sentences are correct is a different way of reasoning than just Bookkeeping alone. Rather, this idea incorporates ideas of Relativity.

Similar to the contexts and student responses illustrated by Whitacre et al. (2012b, 2014), other researchers have utilized similar bookkeeping contexts with debts and assets. Mukopadhyay, Resnick, and Schauble (1990) posed a context with debts and assets in the form a multi-part story, which was an extended narrative called, “The Story of Sam.” In this story, Sam encounters various scenarios of borrowing and gaining money. And, the students reason about this singular changing quantity step by step through the story. A distinguishing feature of a context that supports Bookkeeping is when there is one quantity, rather than two, that is being changed. Mukopadhyay et al. (1990) found that this context of debts and assets with this singular changing quantity context promoted misconceptions, where a DNL, Divided Number Line, mental model (Peled et al., 1989) was re-enforced or promoted.

Poirier and Bednarz (1991) and Ulrich (2013) highlighted reasoning about signed numbers as pointing to “directed change.” Ulrich (2013) wrote, “every signed quantity can be thought of as change in quantity” (p. 128). Thompson (2013), reflecting on Ulrich, wrote that signed quantities representing directed changed really produces an “arithmetic of vectors is conceptualized as an arithmetic of equivalence classes” where simply summing or adding is different from recognizing a change from A to B (p. 144). Ulrich (2012) reported on a teaching experiment conducted with four middle school students were given a list of weights from each week. Students were asked to find the week that had the biggest change. Similar to Whitacre et al. (2012b), Ulrich thinks that spontaneous use of negative notation indicates sophistication in mathematical thought; however Ulrich points to how this notation may lead to a greater awareness of "differences" rather than negative integers. Another teaching experiment Ulrich conducted included a card game that was played several rounds. Students would keep track how the amount of points they won by or the amount of points they lost by. Ulrich notes that many students began to use the notation "winning by" or "losing by" in front of their running total scores. However Ulrich hypothesizes and points toward notions of Counterbalance when she stated, "that this increased attention to 0 and additive inverses is what allows the construction of a true signed quantity that is not divided into separate positive and negative worlds" (p. 137). Although Ulrich's work support thinking about the integers as gains and losses, or changes in quantity, there is not a distinction between quantities that are gained or lost and quantities that are gained and lost with neutralization. Counterbalance, while also drawing upon changes in quantity and gains/losses of quantities, has a distinguishing factor of neutralization. Although research supports Bookkeeping, much of the research in supporting Bookkeeping, like this, often connects to CMIAS, particularly Counterbalance. For example, although Liebeck (1990) and Stephan & Akuyz (2012) used assets and debts in their research, their work connects well to Counterbalance because of ideas of neutralization and will be discussed in the next section. Table 6 highlights some contexts that may support thinking with Bookkeeping.

Table 6

Some Contexts that Support Bookkeeping

Context	Sample References
Borrowing/Paying Money	Bell (1984); Whitacre et al. (2012a, 2014b)
Wanting/Gaining Baseball Cards	Wessman-Enzinger & Mooney (2014)
Borrowing/Paying of Candy Bars	Wessman-Eninzger & Mooney (2014)

Research with prescribed contexts informing Counterbalance. Counterbalance was one of the least utilized CMIAS from students that posed stories for integer addition and subtraction open number sentences (Wessman-Enzinger & Mooney, 2014). Counterbalance is utilized when students use the integers as neutralizing each other. The zero in Counterbalance represents neutralization. Students often used Counterbalance with unconventional contexts, such as good and bad deeds (Wessman-Enzinger & Mooney, 2014). Ideas of Counterbalance are supported in the literature with pedagogical models like the chip model (e.g., Gallardo, 1994), algebra tiles (e.g., Gallardo & Hernandez, 2005), and contexts, like electron charge (e.g., Battista, 1983).
Electron charge is a context that supports reasoning about the integers with Counterbalance (e.g., Battista, 1983). With electron charges, the protons are represented with positive integers and the electrons are represented with electrons. Battista (1983) suggested of conceptualizing the integers as a “collections of charges.” Within this model, each positive charge “cancels” the negative charge or every proton neutralizes the charge of every electron. A unique aspect of the both the counterbalance model in general and the electron charge conceptual model is that each integer or charge has an infinite

amount of representations. For example, consider the integer +5. This can be represented with five protons and no electrons, or 5 + 0. Or, the charge +5 can be represented by six protons and one electrons, or 6 + -1 = 5. Similarly, the charge +5 can also be represented by seven protons and two electrons, or 7 + -2 = 5. Others have discussed electron charge as a potential context for integer instruction (e.g., Frand & Granville, 1978; Grady, 1978); however, Battista has become a seminal piece on electron charge that many cite. This is perhaps because Battista described all four operations with integers in relationship to this electron context. For example, Battista explained that -3 multiplied by -2 may be conceptualized as removing or repeated subtraction of two electrons or an electrical charge of -2. By removing two electrons three times, the overall electrical charge would increase by +6.

Although electrical charge is a typical context that is associated with counterbalance, there are other contexts that support thinking with Counterbalance. For example, thirty-three children's conceptions of integers situated in grades K-5 were analyzed in the context of happy and sad thoughts, which promotes similar ideas of neutralization as electoral charge. The following contextual problem, which supports Counterbalance thinking, was provided to students:

> Everyday Jessica has happy thought and sad thoughts. If she has one happy thought and one sad thought, then she just feels normal-not happy or sad. On Monday, Jessica had 2 happy thoughts and 7 sad thoughts. What kind of day was Monday? (Whitacre et al., 2012a, p. 358).

Along with this task, discrete objects in the form of circles with a smile or frown contained inside were presented to represent the quantities of happy and sad thoughts were provided in the problem. Before the implementation of this context for the aforementioned study with children, this context originally emerged in a student's explanation in pilot interviews about negative integers. For all thirty-three children in the study, reasoning for evaluating the happiness or sadness of a day was exhibited in three different ways: a sign function, a balance metric, or an explicit integer sum. Sign function reasoning occurs when the child refers to a day as happy or sad based on a comparison of the whether a day is composed of more sad days or more happy days. Whitacre et al. (2012a) compared this reasoning to algebraic sign function where the function only takes three possible values: -1, +1, or 0. Similar to this comparison, the children utilizing this reasoning view the days as happy, sad, or neutral. With balanced metric reasoning, the children treat the happy and sad days as canceling in pairs. For the children reasoning in this way a happy day and a sad day "cancel" and the balance remaining is examined. Explicit integer sum reasoning occurs when the children interpret a happy day as a positive integer and a sad day as negative integers. The sum of the situation is computed and the sign of the sum is interpreted as a happy or sad day. Whitacre et al. (2012a) consider explicit integer sums as a more sophisticated way of reasoning since, "The ability to 'see integers' in story-problem or real-world contexts is a characteristic of individuals with sophisticated understanding of integers" (p. 364). This study provided support that children have nascent ideas about integers situated in a context that is approachable. Other researchers have used other contexts that also support using Counterbalance, like voting and good/bad deeds (Shore, 2005).

Table 7

Some Contexts that Support Counterbalance

Context	Sample References
Electron charge	Battista (1983); Frand & Granville (1978); Grady (1978)
Happy/Sad Thoughts	Whitacre et al. (2012a)
Good/Bad Deeds	Shore (2005); Wessman-Enzinger & Mooney (2014)
Boys/Girls Dancing	Dienes (2000)
Balloons	Janvier (1985); Reeves & Webb (2004); Lamb & Thanheiser (2006)
Card Games with Integer Cards	Sasaki (1993); Wessman-Enzinger & Bofferding (2014); Bofferding & Wessman-Enzinger (2015)
Weights	Orlov (1971)

Although the possibilities for contexts that support Counterbalance are infinite, Table 7 highlights some of the contexts that promote the counterbalance conceptual model in the literature. Most of the literature highlighted in Table 7 discusses the instructional experiences and focuses on descriptions of the activities, rather than on the student thinking that was utilized when making use of these contexts.

However, contexts that may support Counterbalance can be innocently implicit. For example, one may reason contexts such as money may always be bookkeeping. However, Stephan and Akyuz (2012) recently conducted a teaching experiment in a 7th-grade classroom to develop a hypothetical learning trajectory where through a financial situation counterbalance ideas were promoted. Situated in a financial context through a

RME lens, the children developed concepts of integer addition and subtraction through their previous knowledge of assets, debts, and net worth. This context is similar to Whitacre et al. (2012b) in the sense that it also utilized money. However, the students in the Stephan and Akyuz teaching experiment provided abundant evidence of using and learning about negative integers with money. Although Stephan and Akyuz (2012) incorporated the use of money as a context, their context allowed for more opportunity for the negative integers to emerge. The reason that the student productively reasoned with money with negative integers in this context is because Stephan and Akyuz modified the typically debt and assets context to incorporate "net worth." Rather than just using the integers as gains and losses, like a bookkeeping perspective, the integers were debts and assets that counterbalanced each other to provide one's net worth. Even in the context of money, students can reason with a counterbalance conceptual model for integers. Stephan and Akyuz (2012) investigated a variety of topics within this net worth context, including the absence of commutativity with subtraction.

Similar to Stephan and Akyuz (2012), Liebeck (1990) also incorporated a counterbalance perspective in her study by using a game called *Scores and Forfeits* In this game, the students would use colored counters as manipulatives. The scores were black counters and the forfeits were red counters. The children, in this game, used the counters to solve integer addition and subtractioon problems and write number sentences for the situations. Liebeck used this *Scores and Forfeits* game, which promoted counterbalance thinking, as part of an experiment to compare children's reasoning in the *Scores and Forfeits* context with children's reasoning in a context of moving along number line. This second component of Liebeck's study that utilized the number line, promoted the translation conceptual model, which will be discussed in the following section, “Research informing Translation.”

Contexts are not the only way to promote thinking with Counterbalance. The chip model, a prevalent manipulative used during instruction with integer operations, supports the Counterbalance conceptual model (e.g., Flores, 2008; Vig, Murray, & Star, 2014). Often in the chip model, positive integers are presented with one color (e.g., black chips) and negative integers are represented by another color (e.g., red chips). For example, $-2 + 3$ could be presented by two red chips and three black chips. A key characteristic of the chip model is identifying “zero pairs.” That is, since $1 + -1 = 0$. There are two sets of zero pairs in $-2 +3$ because we could think of this $(-1 + -1) + (1 + 1 + 1) = (-1 + 1) + (-1 + 1) + 1 = 0 + 0 + 1 = 1$. The chip model is complicated when we have to add in zero pairs unnaturally to solve problems like $-1 – 3$ (Vig, Murray, Star, & 2014). As Vig, Murray, and Star (2014) shared, not all models, like the chip model, work well for all integer addition and subtraction problems. Lytle (1994) considered the chip model an intuitive model for children, but reflected on a way that this model broke down for students:

When asked to give the result of the subtraction $-4 - -6$, S1 remarked, “and you're left with positive 2. But we're not doing that, case these weren't positives,” giving indication that although he had learned a procedure for subtraction of chips, he was not able to intuitively be convinced that the result was valid as the result of the written integer problem. (p. 197)

In addition to the chip model, double abaci and colored beads on abaci can serve as a way to represent addition and subtraction of integers with neutralization (e.g., Koukkoufis & Williams, 2006; Linchevski & Williams, 1996). Similarly, algebra tiles can be utilized in a similar way (e.g., Schorr & Alston, 1999).

Research with prescribed contexts informing Translation. Translation was a conceptual model that students frequently used in my pilot study (see, e.g., Chapter III), but was not used much with the eight-graders that I worked with (Wessman-Enzinger & Mooney, 2014). A mathematical translation is a function that moves every point in a domain a fixed distance. The movement of that function can be considered a directional vector, which has directed magnitude (length). Herbst (1997) discussed the use of the number line metaphor as a way to make sense of integer addition and subtraction. Similarly, Lakoff and Núñez's (2000) identification of order as a foundational component of mathematical cognition and arithmetic as "motion along a path" as one of their four "grounding metaphors" (p. 21) supports and informs Translation. For example, both Herbst (1997) and Lakoff and Núñez (2000) suggested that negative numbers are constructed as point locations within this motion metaphor, using the idea of symmetry on the number line. Thinking about the addition or subtraction of integers as Translation, or with a metaphor for movement, is a common pedagogical approach to teaching the integers (e.g., Nurnberger-Haag, 2007; Tillema, 2012). However, most of the research literature talks about *transformations* of the integers, rather than translations (Marthe, 1979; Thompson & Dreyfus, 1988; Vergnaud, 1982).

Mathematical transformation can be defined as a function over a domain that maps all of those elements to the range. Transformations include, but are not limited to, dilations, reflections, rotations, and translations. I would argue that because the research of Marthe (1979), Thomspon & Dreyfus (1988), and Vergnaud (1982) is situated in

student thinking about the addition and subtraction of integers, they used translations rather than transformations. Although a translation is transformation, not all transformations are translations. Distinguishing their research about student thinking about addition and subtraction as informing Translation instead transformations provides some specificity, especially when we think prospectively to what research on student thinking about multiplication might look like, where transformation will more than like have a different meaning (e.g., negative integers as both scalars and vectors).
While research has pointed to Translation as a way to think about integer addition and subtraction (e.g., Wheeler, Pearla, Bell, & Gattenga, 1981), other researchers, like Marthe (1979) and Vergnaud (1982), have provided problem types that support Translation as well. Bell, Marthe, and Vergnaud, pointed to thinking about integer addition and subtraction as a static number or initial starting point, a translation, and then a static number or final ending point. The work of Bishop et al. (2014) supports this work with these findings with the ways that young students solve integer addition problems. Bishop et al. (2014b) shared that students in their study solved integer problems: "Starting point + Change = Ending Point."

For informing Translation, it is important to understand the contextual problem types that may support those ways of thinking. Marthe (1979), in the first paper about negative integers in PME proceedings, classified different problem types for additive structures for integers. The first category that Marthe described was S_iTS_f, where the initial state (S_i) is translated (T) to the final state (S_f). Marthe then described that either S_i, T, or S_f could be the unknowns in any given problem. A second category Marthe described was $T_1T_2T_3$. He described T_1, T_2, and T_3 as "transformations," although they can also be described as linear translations. From this problem type, Marthe described that there are three subsequent problems that can be posed, where T_1, T_2, or T_3 are unknowns, and T_1, T_2, or T_3 have differing magnitudes and signs. Marthe provided contextual examples of each of these problems. For example, for the problem type $T_1T_2T_3$ with T_2 unknown, T_1 and T_3 with opposite signs, and $|T_1| < |T_3|$, Marthe provided the example, "A car makes an initial journey of 20 km upstream. Then it makes a second journey. If it had made only one journey from its starting-point to its destination, it would have made a journey of 25 km downstream. Describe the second journey" (p. 156). Marthe also included a category, SSS, which is composed of all states and no translations. Potentially, SSS could be a problem type in Bookkeeping or Counterbalance. Marthe stated that this problem type is more challenging than STS. Similarly, in terms of Marthe's problem type STS, Vergnaud (1982) pointed that the minus sign can illustrate a direct transformation, or translation, or the minus sign can represent the inversion of a direct transformation, or translation, which is more challenging. The "minus sign" is used for finding differences; yet, the plus sign can also mean a difference between two directed numbers of different signs. Vergnaud provided "$x + (+4) = -3$, $x = (-3) - (+4) = -(3 + 4)$" for example (p. 73). Vergnaud (1982) also stated, "My view is that equalities and equations do not fit equally well all situations met and handled by students, but only a few of them" (p. 74). Vergnaud was stating that not all equations or number sentences fit contextual situations equally. In terms of Translation, Vergnaud made an important distinction that thinking about moving backwards two units from one, may be represented by both the expression $1 – 2$ or $1 + -2$; however, each of the expressions may not *conceptually* represent this situation equally.

Temperature may be a productive context for connecting integer operations to Translation (Alitparmak & Özdoğan, 2009; Beatty, 2010). Wessman-Enzinger and Tobias (2015), using the context of temperature, modified the Marthe (1979) problem types to include a distinction between directed distance and undirected distance, with SST and SSD respectively. These are the more challenging problems that Vergnaud (1982) pointed to. When a problem is posed with two given relative numbers and the translation is unknown, it is classified as an SST problem. Whereas, when a problem with two numbers and a distance, without direction, it was considered to be an SSD problem. Although not mathematically correct, a preservice teacher provided the following SST problem for the number sentence -17 + 12 = □.

> It was 12° outside Wednesday. It was 17 below zero degrees Thursday. How much had the temperature dropped since Wednesday?

The distinguishing feature of the SSD problem type is that no direction is provided in the problem. For example, a preservice teacher provided the following SSD problem for the number sentence -14 – -20 = □.

> One day in New York it is -14 degrees out. In Maine the same day it was -20 degrees. What is the difference between the two states' temperatures?

Wessman-Enzinger and Tobias (2015) also argued that the SSS problem type is not an appropriate problem type for Translation. These modified problem types are summarized in Table 8 below.

Table 8

Translation Problem Types from Wessman-Enzinger & Tobias (2015)

Problem Type	Description
STS	A problem posed with a relative number and a translation, with the second relative number as the unknown, is considered to be a STS problem.
TTT	A problem posed with two given translations and the third translation is unknown is considered to be a TTT problem.
SST	A problem posed with two given relative numbers and the translation is unknown is considered to be a SST problem.
SSD	A problem posed with two relative numbers and a distance, without specified direction, is considered to be a SSD problem.

Selter, Prediger, Nührenbörger, Hußmann (2012) differentiated between the "taking away" and "determining the difference models" of subtraction. If, we consider these models, take away and difference, it seems as if the take away model would be closely related to Bookkeeping and Counterbalance, whereas the difference model would be closely related to Translation and Relativity. It seems as if the distinction between SST and SSD provides further insight into the difference model of subtraction, with one representing a directed distance and the other an undirected distance. Understanding STS, SST, and SSD also seems more related to the difference model of subtraction, with distinction of directed and undirected difference.

Thompson and Dreyfus (1988) provided a rich instructional context in a mircoworld, called INTEGERS, for two sixth graders. Within this microworld, the students solved contextual problems that were often of the problem type TTT. That is, students constructed two different translations of a turtle and determined the net translation of the turtle. Thompson and Dreyfus worked with two sixth graders on these translations for 6 weeks. Using a person instead of turtle, Liebeck (1990), similar to Dreyfus (1988), included a number line with a person that moved along this number line. Liebeck's activity also differed because it did not incorporate visualized directed vectors and there was only one movement. Rather, the students would start at different points, such as 2 or -5, translate the person from that point, and then find the ending point. Addition and subtraction of integers was described as "when we add we move forwards" and "when we take away we move backwards." Liebeck even had a table for students to record the starting place, moving forwards or backwards, the ending place, and then the "answer" or the number sentence. This use of the person moving on the number line, related to Marthe's (1979) problem type, STS. The contexts of Thompson and Dreyfus (1988) and Liebeck provide support for thinking in Translation, but there are many other contexts that have been utilized that may support Translation (see Table 9).

Table 9

Some Contexts that Support Translation

Context	Sample References
Timeline with BC and AD dates	Gallardo (2003)
Temperature Increasing/Decreasing	Wessman-Enzinger & Tobias (2015)
Travelling up and down a river	Marthe (1979)
Riding in an elevator	Iannone & Cockburn (2006); Larsen & Saldanha (2006)
Balloons moving up and down	Janvier (1985); Reeves & Webb (2004)

Although contexts can support thinking about Translation, a typical pedagogical tool that support Translation is the number line. For contemporary mathematics educators, the number line often serves as a key pedagogical tool for representing real numbers (e.g., Saxe, Diakow, & Gearhart, 2013) and can support the understanding of the translation conceptual model. Educators and researchers argue for the benefits of and the use of number lines with young children (e.g., CCSSO & NGA, 2010; Wu, 2011). The widely-used text *Everyday Mathematics*, went so far as to recommend that number lines be hung on classroom walls in all grades, starting in Kindergarten, as a way to facilitate the learning of negative integers (e.g., Smiddy, 2008).

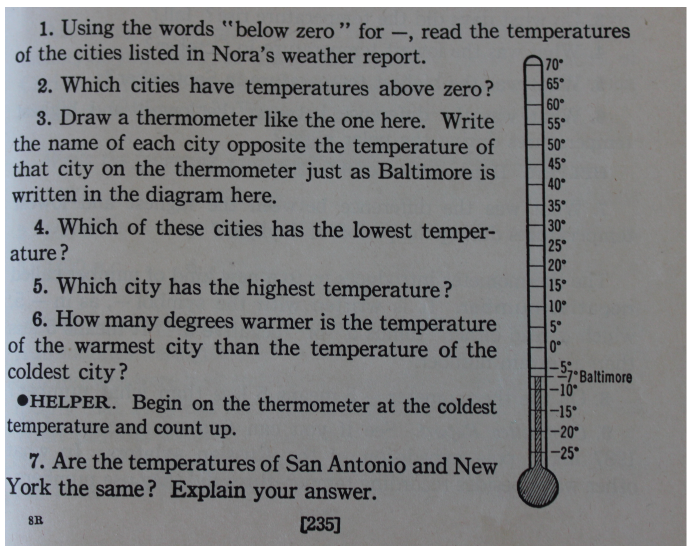
As current research on student thinking about the teaching and learning of integers and specifically negative integers, there can be no doubt that the concept of a number line is foundational not only to informing Translation, but also to informing the current research in mathematics education on student thinking about number, and specifically negative integers (e.g., Bofferding, 2014; Saxe, Diakow, & Gearhart, 2013). Although historical developments of a concept may not parallel psychological developments, a deep understanding of the past can offer educators' perspectives on, and an understanding of, the present and can therefore help them to decide wisely for the future. As Sfard (2008) pointed out that, "one becomes … bewildered when one notices the strange similarity between children's misconceptions and the early historical versions of the concepts" (p. 17). It is interesting, though, to note that very few models other than the number line have been used to investigate the teaching and learning of negative integers. A historical perspective on the evolution of the number line could give insight to the teaching and learning of negative integers (Thomaidis & Tzanakis, 2007; Peled &

Carraher, 2008). Historical research has shown that although some mathematicians had conceived of the number line in the seventeenth and eighteenth centuries (e.g., Wallis, 1685), most educators did not refer to number lines when attempting to make sense of operations negative integers (Heeffer, 2011). Mathematicians during the seventeenth and eighteenth centuries would often make sense of negative integers by using contexts, such as debts of money, or they would incorporate geometrical approaches within explanations on rules of operations with negative integers. Heeffer (2011) has presented historical evidence that mathematicians have struggled in the past to utilize number lines with operations, such as division, in their efforts to make sense of the negative numbers and their operations. Although extensive research has been conducted in the realm of the history of negative numbers with the number line (e.g., Henley, 1999; Schubring, 2005) and research that supports student thinking with respect to the negative numbers and number line (see, e.g., Bishop et al., 2011; Bishop et al., 2014b; Saxe, Diakow, & Gearhart, 2013), specific attention to the intuitive development of the number line with children within the realm of the teaching and learning of integers has lacked attention.

Indeed, research has addressed children's conceptions about the number line and identified challenges that children have. A major challenge that children may have with the number line is that the distance unit between the tic marks is to be utilized, not the tic marks themselves (Carr & Katterns, 1984; Ulrich, 2012). When student count the tic marks, rather than the distances between the tic marks, they will end up with one more (or one less) than anticipated (Barrett et al., 2012). A major assumption with the number line as a pedagogical model for instruction with integer operations, is that it is assumed that students will be able to extend their previous knowledge about whole numbers and the number line to operations with the integers and the number line. Ernest (1985) stated that the "number line model does not have any compelling inner logic. Instead it assumes familiarity with underlying representational conventions, which are to some extent arbitrary" (p. 418). However, the number line can certainly be tool for extending whole number reasoning with integers. For example, students may extend their use a number line with whole number like a number path, incorporating negative integers, instead of making use of the number line as mathematically or culturally expected (e.g., Wessman-Enzinger & Bofferding, 2014). Yet, mathematicians have historically illustrated trouble with connecting all of the integer operations to the number line (Heefer, 2011). Vig, Murray, and Star (2014) discussed the affordances and hindrances of different models, such as the chip model for integer addition and subtraction and the area model for fraction addition and subtraction. The issues of the coordinating of tic marks and spaces on a number line, extending whole numbers to include negative integers, and making sense of all four operations of with the number line as a tool, are certainly some conceptual hurdles for students as they begin to operate with the integers, but there will be affordances and hindrances of all models. The notorious historical struggle of mathematicians merely points to conceptual struggles of using the number line; however, these are not places where the number actually breaks down as a tool for a learner. Reflecting on this, Liebeck (1990) reflected on the hindrance of the number line and stated, "The number line, then, emphasizes ordinality at the expense of cardinality" (p. 237). Liebeck pointed to an idea that some conceptual models, like Translation support ideas of order more than Bookkeeping. And, CMIAS, like Bookkeeping, may support ideas of cardinality more. Liebeck's point is not that the number line is not important or a crucial pedagogical tool. Rather, Liebeck this points to the large conceptual leaps that a child may have to undertake to begin to use the number line with the integer operations and the conceptual challenges they will encounter as they learn to use the number line. And, some CMIAS may support some ways of reasoning more than others.

A plethora of literature connecting the number line with integer operations includes contrived rules to make sense of the integers. For example, Nicodemus (1993) described “Linesman” where a human is standing on a number line facing right, negative number represents facing the opposite direction or walking backwards, and addition and subtraction represent moving forwards or backwards. Herbst (1997) also illustrated these rules that he found in a textbook analysis. For example, when considering the number sentence 2 – -3, it is suggest that one conceptualizes starting at two on the number line, turning around, and walking backwards three space on the number line, getting to five. These types of rules may not be intuitive to children. Making sense of all of the integer operations on the number line may be challenging to students and we need to learn how they make sense of them, rather than telling them how to make sense of them.

Research with prescribed contexts informing Relativity. Relativity was the least utilized conceptual model from students that posed stories for integer addition and subtraction open number sentences (Wessman-Enzinger & Mooney, 2014). Relativity is utilized when students use the integers in comparison to an unknown referent. The zero in Relativity represents an unknown referent or an arbitrarily chosen referent. One of the students that used Relativity with a story about baseball, where one was down runs and then up runs and zero represented a tied baseball game (Wessman-Enzinger & Mooney, 2014). Although students hardly made use of the integers with Relativity, historically, the integers were first presented in North American arithmetic and algebra texts as relative numbers (Wessman-Enzinger, 2015). Other historians have noticed the historical prevalence of mathematicians discussing relativity with integers (Hefendehl-Hebeker, 1991; Herbst, 1997; Hitchcock, 1997). Some of the first illustrations of the number line in North American school mathematics arithmetic texts and algebra texts with a relative number line (see, e.g., Chapter I, Figure 2) and were the temperature scale (see, Figure 6). Temperature represents an important context for not only for use of Translation, but also for Relativity.



1. Using the words "below zero" for –, read the temperatures of the cities listed in Nora's weather report.

2. Which cities have temperatures above zero?

3. Draw a thermometer like the one here. Write the name of each city opposite the temperature of that city on the thermometer just as Baltimore is written in the diagram here.

4. Which of these cities has the lowest temperature?

5. Which city has the highest temperature?

6. How many degrees warmer is the temperature of the warmest city than the temperature of the coldest city?

●HELPER. Begin on the thermometer at the coldest temperature and count up.

7. Are the temperatures of San Antonio and New York the same? Explain your answer.

8B [235]

Figure 6. Temperature Scale in Buswell, Brownell, and Lenore (1938, p. 235).

This idea of Relativity is one that required a long evolution historically. In fact, Hefendehl-Hebeker (1991) reflected on the historical struggles of Euler and Lagrange, "There was no notion of a uniform number line. The preferred model was that of two distinct oppositely oriented half lines. This reinforced the stubborn insistence on the qualitative difference between positive and negative numbers. In other words, these numbers were not viewed as 'relative numbers.'" (p. 27). Hitchcock (1997) wrote a play dramatizing the history of negative numbers. In this dramatization, he portrayed Alfred North Whitehead as proving a relative number line (see, Figure 7).

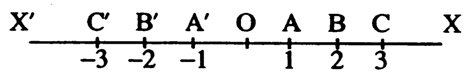


Figure 7. A Relative Number Line from Hitchcock (1997).

The integers were created in Figure 7 in relationship to the point O. Points A, B, and C, which are to the right of O were considered positive numbers an points, A', B', and C', which are to the left of O were considered negative, although it did not have to be that way. Certainly A', B', and C' could have been positive numbers and A, B, and C could have been negative numbers.

Our modern mathematical definition of integers, which includes the integers as a subset of the real numbers and rational numbers, assumes the integers as objects. Our mathematical definition of integers doesn't highlight the imperative use of integers as relative numbers; and, our standard documents (e.g., National Council of Teachers of Mathematics, 2000; CCSSO & NGA, 2010) do not highlight the relativity of integers as well. As mentioned prior in this literature review, Gallardo (2002) points to one understanding of integers as a number is as a relative number. Carraher, Schliemann, and Brizuela (2001) reflected on an N-number line, where the ordering is centered around N (e.g., N – 3, N – 2, N – 1, N, N + 1, N + 2, N +3). A distinguishing element of this N-number line is that N is unknown and could be represented by any number. The N-number line presented by Carraher et al. captures the essence of "relative numbers" and "relative number lines" found in early arithmetic and algebra texts in the nineteenth century. For example, integers as relative numbers is something that mathematicians valued as they first grabbled with understanding the integers (see, e.g., Durrell &Robbins (1897); Loomis, 1857). For example, Loomis (1857) began his introduction of the negative integers with the following:

> The term subtraction, it will be perceived, is used in a more general sense in algebra than in arithmetic. In arithmetic, where all quantities are regarded as positive, a number is always *diminished* by subtraction. But in algebra, the difference between two quantities may be numerically greater than either. Thus, the difference between $+a$ and $-b$ is $a + b$. The distinction between positive and negative quantities may be illustrated by the scale of a thermometer. The degrees above zero are considered positive, and those below zero negative. From five degrees above zero to five degrees below zero, the number stand thus: +5, +4, +3, +2, +1, 0, -1, -2, -3, -4, -5. The difference between give degree above zero and give degrees below zero is ten degrees, which is numerically the *sum* of the two quantities. (p. 17)

In this excerpt, the description included the order of the negative integers through the context of the thermometer. After discussing the thermometer and ordering, Loomis transitioned and commented on relativity:

> It has already been remarked, in Art. 5, that algebra differs from arithmetic in the use of negative quantities, and it is important that the beginner should obtain clear ideas of their nature. In many cases, the terms positive and negative are merely relative. They indicated some sort of *opposition* between two classes of quantities, such that if one class should be added, the other ought to be subtracted. Thus, if a ship sails alternately northward and southward, and the motion in one direction is

> called positive, and the motion in the opposite direction should be considered negative. (pp. 18–19)

In this description the integers are described as a relative number. Although the context of thermometer can be used to discuss ideas of translation, thermometers and temperature can also get ideas of relativity. The idea of relativity is intimately related to ideas of relativity in the creation of the temperature scale. For example, consider the temperature scale in in Figure 6. Guy Buswell, William Brownell, and Lenore John (1938), co-authors of the arithmetic series, *Living Arithmetic*, a popular arithmetic book series of the early and mid-twentieth century, included negative integers and an integer number line in a text intended for students of Grade 8. Notice that Buswell, Brownell, and Lenore first described negative integers with a thermometer, and included an illustration of thermometer, similar to a number line; however, there is not a scale defined. Was the scale in this text the Russian scale (commonly mentioned during this time frame), Celsius, Kelvin, or the Fahrenheit scale? These scales were created, although intentionally, arbitrarily. The temperature scale represents scientists' relative decision to make a certain degree zero. That is, the negative temperatures represent the relative temperature in relationship to zero. Notice in this description and illustration of the thermometer, what temperature scales were utilized is not identified. And, the zeroes in each of the aforementioned scales are not equivalent. The German physicist, Daniel Gabriel Fahrenheit, determined the zero of his scale differently than how the Swedish astronomer, Anders Celsius, determined the zero of his scale. However, even though their scales of measurement differ and 0°C does not equal 0°F, there exists equivalence temperatures. We know that 0°C equals 32°F. This subtle nuance of integers as relative numbers needs to be highlighted with students and investigated as they use relative numbers when operating.

Herbst (1997) reflected on how the translation conceptual model is related to the relativity conceptual model. He wrote, “The statement of an addition of the number line involves the juxtaposition of two arrows, a relative position” (p. 38). Similarly, Marthe (1982) utilized a river problem for investigating the thinking and learning of integer addition and subtraction. In this problem, the positive integers represented moving upstream the river and the negative integers represented moving downstream the river. This is upstream and downstream is all relative to the initial starting point on the river. Wherever one starts at on the river, represents the zero. Exactly where one starts at this river is unknown; yet, everything is measured from this point. Brodie (2007) prompted students to unpack what zero meant. Because within the relativity conceptual model establishing what zero is has an infinite amount of possibilities, this may be challenging for students.

Interestingly, the idea of relativity is a lost nuance about integers, with little attention in research about students’ thinking and learning. Gallardo (2002) is one of the few researchers that has investigated thinking about integers in relationship to integers as a relative number; whereas, most of the research discussing relativity is situated in historical sources and investigations. Ulrich (2012) also hinted at relativity when she reflected on increases and decreases, similar to Bookkeeping, without a reference point:

> However, the situation is less intuitive when some of the signed values represent positions: Recognizing that starting at +3 and decreasing 7 gets you to the same value as starting at -7 and increasing 3 requires greater reflection on the situation.

> In addition, when working with changes, you can have an unspecified reference point. When students eventually work with vectors and matrices, the ability to deal with an unknown reference point is crucial. In addition, when interpreting slopes and rates of change in functions, the actual values of the underlying quantity are not as important as the way in which those values change. (p. 25)

In a footnote, Ulrich (2012) provided another example for relativity, "By *unspecified reference point*, I mean that you know what relationships the reference point has to the changes in quantity, but its value in terms of the underlying quantity is unknown. For example, if you put 30 cents in a piggy bank and take out 40 cents, you can figure out the overall change in the piggy bank's value without knowing how much was in it originally" (p. 25). This unspecified reference point, is the distinguishing feature of the Relativity. Although Table 10 illustrates contexts that may be used for the development of Relativity, there is a large gap in the literature of descriptions of student thinking and use of Relativity.

Table 10

Some Contexts that Support Relativity

Context	Sample References
Temperature Scale	Wessman-Enzinger & Tobias (2015)
Up/down runs in baseball without known score	Wessman-Enzinger & Mooney (2014)
Increases/decreases of money in piggy bank	Ulrich (2012)
Getting on and off a train with unknown amount of riders	Bishop et al. (2010)

Rule as an Important, but not a Solitary CMIAS

Bell (1983) conducted a study with 25 students who were fifteen years old. Most of the students that Bell worked with relied on rules rather than understanding. Students had a tendency to ignore signs of integers and they combined the magnitudes by referring to the operation sign. Bell, O'Brien, and Shiu (1980) reflected on students' conceptions about integers:

> In directed numbers we are developing teaching experiments based on the hypothesis that deep-lying difficulties which many pupils experience are due to an inadequately connected conceptual structure; that is, the provision of rules for coming the numbers themselves is connected perhaps too weakly even with the number line and the co-orindate plane, and hardly at all with other phenomena such as bank transactions and balances, journeys northwards and southwards on a motorway, fast and slow clocks, and so on." (p. 121)

Bell points to the phenomena that students do not connect the negatives to conceptual underpinnings, like the number line, and students do not readily connect the integers to conventional contexts. Rather, students typically reason with rules about the integers. Perhaps this is because Bell's students were participants that are older than some of the participants in other studies (e.g., Bofferding, 2014; Bishop et al., 2014a, 2014b). Or, this could be evidence that students prefer to reason with the integers with Rule. In fact, in my previous study, many students used Rule more often than the Translation, Counterbalance, and Relativity (Wessman-Enzinger & Mooney, 2014). And, Bofferding & Richardson (2013) demonstrated that many preservice teachers often use Rule, as they primarily connected integer operations to rules about whole numbers.

Vlassis (2008) reflected on the role of the minus sign as unary, binary, and symmetric in relationship to the formal rules for operations with negative numbers that students employ. Vlassis reflected:

> Students' difficulties can no longer be summed up as the obstacle of the concept of negative numbers as abstract mathematical objects but rather relate to the use of symbols and in particular of the minus sign. The capacity to take account, according to the context, of the unary, binary and symmetric dimensions and to display considerable flexibility in doing so is vital to students' ability to make sense of these numbers which, above all, obey various formal rules. (p. 569)

An important insight Vlassis provides here is that understanding the negatives is more than just making an abstraction and just learning various formal rules. For example, Ryan Williams, and Doig (1998) found that students were often confounding their rules for integers. That is, they often reasoned that -2 – -8 was 10 or -10 because they were combining a rule that 2 + 8 = 10 with "minus a minus is a plus."

Part of the complexity of investigating student thinking about addition and subtraction of integers is that many things, such as order and the role of the minus sign, influence the learning of operating with the integers. For example, students understanding about the uses of negative numbers (Gallardo, 2002) may influence how they operate with the negative numbers. Or, students' understanding about order and directed magnitude (Bofferding, 2014) may influence the rules that develop or use when learning to operate. Or, the CMIAS students typically utilize and the contexts that they experience (Wessman-Enzinger & Mooney, 2014) may influence their understandings about the rules that students have and apply to integers. Nonetheless, there is evidence that students draw upon rules still (e.g., Bofferding & Richardson, 2013; Bishop et al., 2014) and there is push by the CCSSO & NGA (2010) to make sure that students learn to operate. And, of course, operating accurately is important (Lamb et al., 2013). Yet, we know that children often develop rules about integers that are not true. For example, Gallardo (1994) discussed that because students in her study interpreted the minus sign only as a subtraction symbol they thought that the negative in front of the b in -a – -b (where a and b > 0) is not necessary. Gallardo and Hernandez (2005), also found that for problems like students thought that -8 – 7 = -1. But, sometimes the students in Gallardo's (1994) study utilized rules that were correct. Her students correctly interpreted -a – b (where a and b > 0) as always negative. Because procedural fluency (Kilpatrick, Swafford, & Findell, 2001) is an important component to learning to operate with numbers, we need to make better sense of what rules students use and how the develop these rules conceptually. Marthe (1979) reflected about learning the integers, "In a class, if the goal is not to obtain the repetition of algorithms by children, but if it is established as the acquisition and development of genuine knowledge, the children must be presented problems rich in content, so that they can assimilate the concepts" (p. 323). As Marthe highlighted, we want our students go beyond learning just the rules of integers. Three decades later, Woo (2007) reflected on similar reflections about the teaching and learning of operations with negatives:

> The instruction of negative number in middle school ends with incomplete, complex models. Students' understanding of computational principles of negative number is very low, and they end up memorizing rules, thus accept (-1)(-1) = 1 without any impressive moments. Despite the limits of models used in school,

> students, teachers, and even the writers of textbooks cannot go far enough to reconsider the origin of the problem, and lack clear understanding of formal essence of the negative number. (p. 83)

If we wish to help students understand negatives and the formal essence of the negative numbers, we need to learn more about how their rules for operations are developed in relationship to the CMIAS.

Identifying the learning of operations in relationship to the CMIAS, and not just focusing on operations alone is pivotal because learning about the integers transcends operations alone. Hativa and Cohen (1995) distinguished "operation sense" and "number sense" with negative integers. Similarly, Kilhamn (2009) theorized about what number sense is in relation to concepts involving integers. These components include intuitions about numbers and arithmetic" (p. 331), the "ability to make numerical magnitude comparisons" (p. 332), the "ability to recognize benchmark numbers and number patterns" (p. 333), and "possessing knowledge of the effects of operations on numbers" (p. 334). By making some components of number sense explicit for the teaching and learning of integers, she implicitly points to the notion that the transition from whole number reasoning to integer reasoning is not sufficiently accomplished in learning operations alone. Lamb et al. (2013) presented "Integer Sense" at PME-NA 2013 Integer Working Group. As part of the Integer Sense described there, it was recommended that:

> Student exhibit multiple conceptions of integers. That is, they conceive of integers as inverses, locations and directed movements on a number line, directed magnitudes, embodiments of real-world phenomena, and members of an

equivalence class. They may use a variety of tools for making of integer-related problems. (Lamb et al., 2013, p. 1361)

As a field we seem to agree that learning about the integers entails more than operations and learning rules for those operations. Rather, there are larger mathematical ideas, like those described by Kilhamn (2009) and Lamb et al. (2013) and illustrated in the descriptions of the CMIAS. The CMIAS point to ways of thinking about and using integers with conceptual understanding beyond operations alone.

Developmental Perspectives in the Integer Literature

Present in the literature are strategies that students may use when reasoning about integer operations (e.g., Human & Murray, 1987). There are also descriptions broad ways of describing thinking: WoR, Mental Models, and CMIAS that these strategies may emerge out of or be produced from (Bofferding, 2014; Bishop et al., 2014a; Wessman-Enzinger & Mooney, 2014). A significant gap in the literature is a developmental perspective of student thinking about integer operations. That is, we lack ways of describing how thinking and learning evolves over time. Although Bishop et al. (2014) conducted a cross-sectional study, which shares thinking across various age groups, if the integers and operations with them are interpreted as secondary intuitions (Fischbein, 1987), then cross-sectional studies do not provide insight into the development, or conceptual change, over time because learning is dependent upon instructional experiences with integers. Bofferding (2014) captured development by describing conceptual change by implementing pre- and post- assessments around instructional interventions. In this sense, the assessment captured the conceptual changes, or learning and development, which occurred with the aid of instruction.

Building upon the idea that development of integers is dependent upon instruction (Bofferding, 2014; Fishbein, 1987), learning must be described as a process where these conceptual changes are described and these conceptual changes constitute learning. For this reason, an affordance of commognitive theory (Sfard, 2008) as the guiding lens of this study is that it serves as a useful tool to define and examine learning. Examining learning over time provides an illustration of the development of thinking about integer addition and subtraction by capturing the conceptual changes.

Learning in this study is defined as a change in mathematical discourse (Sfard, 2008). With commognitive theory, learning is defined as a "process of changing one's discursive ways in a certain well-defined manner" (Sfard & Avigail, 2006, p. 4). Thus, identifying changes in students' mathematical discourse is evidence of their learning. Sfard (2008) considers thinking as a communication with oneself and recognizes that learning experiences influence this thinking as communication with oneself. With a commognitve theoretical perspective, thinking mathematically is mathematical discourse. By describing the mathematical discourse, mathematical thinking is described. Because learning is defined as a change of discourse, this study aimed to provide extended time with students in order to investigate the changes in mathematical thinking, or mathematical discourse. Sfard points to the main tenets of mathematical discourse: word use, visual mediators, narrative, and routines.

Word Use

Sfard (2008) classifies a discourse as mathematical if the discourse includes language that is mathematical. Within this study, students' mathematical word use about integers will be examined, paired with other tenets of discourse, to describe their thinking. Examining word use will provide evidence of different uses of CMIAS or insight into things they may draw.

Visual Mediators

Discourses are often focused about a medium, a concrete object, or artifact. As a part of mathematical discourse, visual mediators are produced (Sfard, 2008). Visual mediators with integers may be the mathematical symbols written by students or the drawings they produce to discuss their thinking or solve a mathematical problem. Sfard and Avigail (2006) state that these visual mediators are “part and parcel in the act of communication, and thus of the cognitive processes themselves” (p. 7). The drawings that the students produced in this study are considered in relationship to word use, as well. The visual mediators are a crucial component to examining the mathematical discourse and change in mathematical discourse, even if visual mediators are not part of the spoken discourse. For example, students drew empty number lines and tallies as ways to solve problems. Identifying and analyzing these types of visual mediators is an essential component to making sense of the mathematical discourse about integers.

Narratives

Sfard (2008) defines a narrative as, “a series of utterances, spoken or written, that is framed as a description of objects, or processes with or by objects and is subject to endorsement or rejection, that is, being labeled as ‘true’ or ‘false’” (p. 300). She also defines utterances as, “communicational act in language (this category includes written communicational acts long with the spoken ones)” (p. 302). Thus, the interpretation of narratives in this study is that the narratives will be uncovered by using the written text (i.e., visual mediators) produced by the students and the spoken words (i.e., word use).

Sfard (2008) described students' narratives as including, but not limited to, mathematical definitions, theories, theorems, and properties formed as student interact with the integers. For this reason, narratives can be endorsed or rejected. That is, a student may develop a narrative that is rejected later. Because students' are learning how to operate with integers, their narratives are being generated over time, changing, and will not be stated as "mathematical theories" as they are often done in "scholarly mathematical discourse" (Sfard, 2008, p. 134). Rather, the narratives in this study are the *mathematical uses* that the student employ, as evidenced by their utterances (i.e., word use, visual mediators). In this study, as students develop and make use of various CMIAS, these conceptual models may serve as a way of describing the students' mathematical uses or theory building, otherwise called narratives. The CMIAS, although descriptions of conceptual thinking, are also representations of mathematical uses of integers. That is, the CMIAS describe mathematical conceptualizations, which simultaneously represents thinking *and* mathematical use.

Although conceptual models, the descriptions of the CMIAS also highlight mathematical uses of the integers that are not made as explicit as rules about operations. For example, with Translation the integers are utilized integers as a vector, with Relativity the integers are treated as relative numbers, and with Counterbalance the integers are treated with neutralizations, etc. The CMIAS encapsulate mathematical uses imbedded in the isomorphism of contexts. Commognitive theory also embraces thinking as tenets of discourse, such as mathematical use. Because of this, it is the interpretation of this study that the codes of the various CMIAS, which represent descriptors of mathematical uses of integers, informed by both word use and visual mediators by the students, can be used to describe the students' narratives.

Routines

Routines refer to the set of repetitive patterns in mathematical and nonmathematical activities. This includes the mathematical activity of the participants as they substantiate their mathematical narratives. Sfard (2008) points to the repetitive characteristics of discourse as routines. The idea is that some routines may be inherent and not explicitly communicated as an expectation. Another aspect of routines is identification of when and how the routines occur.

There may be specific routines that students typically draw upon, like a drawing they produce repetitively. Identifying and describing the discursive routines established by the students provides perspective into the overall mathematical discourse. A mathematical routine includes identifying certain patterns. For example, students may utilize draw a number line routinely for solving certain open number sentences more than they do for solving other open number sentences. If students stop drawing particular visual mediators, then identifying their routine and changes in routines is important to describe their learning. These routines are important to analyze and make sense of alongside the other components of commognitive theory to make sense of the students' learning about integers.

Routines also establish repetition. Mathematical discourse is considered, "a collectively implemented activity that, when observed over time in its diverse manifestations, displays repetitiveness, and thus patterns" (Sfard, 2008, p. 195). Since learning is viewed as a change in this discourse, looking at where repetition breaks down can present moments of cognitive conflict. Identifying these and making sense of them points to important aspects of learning about the integers. For example, a break in a routine occurs if a student who typically uses Translation to add then struggles with subtraction. Examining the routines of in the learning of integers is important for this study.

Word use, visual mediators, routines, and narratives comprise four tenets of discourse from a commognitive theory standpoint. Each of the components, although listed separately and described as corpuscles, relates to the other components. For example, word use may work in conjunction with visual mediators, as students intertwine use of these. The students repetitive patterns embedded in this word use and visual mediators that are routine, communicate the students' narratives. Although described separately, these tenets of commognition are synergistic and work together to describe students' mathematical discourses, which is mathematical thinking.

Summary

This chapter opened with a personal reflection of how I came to know and learn about the research on student thinking in the realm of negative integers. Then, this chapter shifted to discussion of research on student thinking about integers situated both in symbolism and context. Specifically, the research described on student thinking situated in symbolism was connected to the CMIAS. And, the research in the domain of context was organized by student-generated contexts and prescribed contexts. Then, the research on student thinking emerging from contexts was discussed around the CMIAS. Organizing the literature this way was done to help identify what we know as a field in relationship to the CMIAS because this dissertation study highlights research that explores student thinking on integer addition and subtraction, which requires use of both in symbolism and contexts. Also, the student thinking that emerged from this dissertation study is used to develop more robust descriptions and modify the existing CMIAS and the development of these is informed by the existing literature. Although the research is often dichotomously situated between reasoning within symbolism and contexts, understanding the integers entails a coordination of both symbolism and contexts. As a field we need to build bridges that connect domains of research, like symbolism and contexts. Furthermore, much of the research on student thinking is either done within instructional interventions or with small amounts of time of individualized interviews per child. Studies that spend extended time with the same children can offer insight to the development of thinking about integers. It is the hope of this dissertation study to develop more robust descriptions of the CMIAS, what CMIAS student use, while also painting a picture of learning about integers over extended time.

CHAPTER III

DESIGN OF STUDY AND METHODS

Methodology

This dissertation study was conducted within the context of a teaching experiment (Steffe & Thompson, 2000). I selected a teaching experiment methodology to investigate students' conceptual models about integers because it provided an opportunity to interact with students for extended time as they learned about integer addition and subtraction. This methodology also provided opportunities for insight into the learning process, which is complex and lengthy. In this study, I defined learning as a change in discourse as described by commognitive theory (Sfard, 2008). The details of the methods of this study, the data, and the data analysis are discussed this chapter.

Participants

Three students were selected from Grade 5. Students from Grade 5 made an ideal selection because they have had mathematical experiences with all four operations with positive integers, or the whole numbers. Also, the recommendations of CCSSO and NGA (2010) in the *Common Core State Standards* indicate that the negative integers are introduced, without operations, in Grade 6. All four operations with integers are recommended for Grade 7 by CCSSO and NGA. According to these recommendations, the teaching and learning of all of the operations with the integers, or with negative numbers, begins and ends within Grade 7. Selecting students in Grade 5 allowed for the instructional experiences in this study to be these students' first instructional experiences with negative integers, while also being as close as possible to the age of the CCSSO & NGA recommendations. A site at a rural elementary school, which used traditional curriculum that did not incorporate negative numbers at earlier grades, was selected.

Two written pre-tests were given to all of the Grade 5 students at the school selection site. The first written pre-test consisted of 10 open number sentences with integer addition and subtraction problems (see Appendix B). Two of the ten addition and subtraction problems used only positive integers and the other eight incorporated one or more negative integers. Students were asked to write a story that they thought matched the number sentence best and solve each open number sentence. The second written pre-test consisted of 8 contextual problems (see Appendix C). The second written pre-test was designed to incorporate two contextual problems for the CMIAS that would be promoted during the Group Sessions (i.e., Bookkeeping, Counterbalance, Translation, Relativity). After each word problem, the students were asked to write a number sentence that they thought matched the given situation, provide a solution, and include any drawings or pictures that they used to solve the problem.

The two written pre-tests were administered over two days to all of the fifth graders at the school. Twenty-five sets of written pre-tests from the fifth graders who volunteered were examined. The pre-test for which students wrote stories for number sentences were coded for use of CMIAS utilized to aid in participant selection. The pre-test where students solved contextual problems, were examined for use of negative numbers. The fifth graders mostly did not use negative numbers when solving these problems. Only a few students used negative numbers for singular problem, which involved a temperature falling below zero. Therefore, the first pre-assessment, where students wrote stories for the number sentences, resulted in serving as the main tool for participant selection.

The selection of participants was influenced by a pilot study that was conducted in May 2013 with 131 Grade 5 students from four different publics schools, both urban and rural, in the Midwest and West. The students in the pilot study were given eight number sentences (e.g., -2 – 3 = -5) and were asked to write stories that matched the eight number sentences (see Appendix A). These number sentences were analyzed for the use of some of the CMIAS (i.e., Bookkeeping, Counterbalance, Relativity, Translation) that were going to be promoted through the use of contexts in this teaching experiment. Counts and percentages were recorded to determine how many of these 131 Grade 5 students used each of these specific conceptual models at least once. Because this teaching experiment provided contexts to try to promote these four of conceptual models specifically, the counts and percentage for these CMIAS are shared. Overall, 121 students (or 92.4% of the students) used Bookkeeping at least once, 6 students (or 4.6% of the students) used Counterbalance at least once, 40 students (or 30.5% of students) used Translation at least once, and 2 (or 0.8%) of students used Relativity at least once. Most students in the pilot study appeared to utilize the Bookkeeping with their stories. Participants were sought out that predominately utilized Bookkeeping. For this study, students that utilized the Bookkeeping along with another CMIAS also seemed important because it represents a potential to think flexibly about the integers upon entering the teaching experiment.

Table 11 illustrates the different CMIAS utilized by the fifth-graders, who were willing to participate in this study, in the first written pre-assessment which was writing stories for open number sentences. This written pre-assessment narrowed the pool of volunteers from 25 volunteer to 15 volunteers, 9 who used Bookkeeping, 1 who used Bookkeeping and Counterbalance, 2 who used Bookkeeping and Rule, 2 who used Bookkeeping and Translation, and 1 Bookkeeping, Translation, and Rule (see Table 11). Of these 15 volunteer students who used Bookkeeping, participants were selected. Because different CMIAS would be promoted in this study, and Translation and Rule are also prominently utilized CMIAS, a student who utilized Translation as well as Bookkeeping was selected and a student who utilized Rule as well as Bookkeeping was also selected. The responses within each of these categories (i.e., Bookkeeping, Bookkeeping and Translation, Bookkeeping and Rule) were examined for the responses that appeared the most detailed. Three participants were selected from each of these categories: Jace, Alice, and Kim, which are pseudonyms. On this written assessment, Jace utilized Bookkeeping and Translation. Alice utilized the Bookkeeping and Rule. And, Kim utilized the Bookkeeping only.

After participant selection was established, these three students (i.e., Jace, Alice, and Kim) participated in the teaching experiment from which data for this study were drawn.

Table 11

Responses from the First Written Pre-Assessment

CMIAS Demonstrated	Number of Grade 5 Students (Percentage of Students)
Bookkeeping	9 (36%)
Translation	3 (12%)
Rule	7 (28%)
Bookkeeping & Counterbalance	1 (4%)
Bookkeeping & Rule	2 (8%)
Bookkeeping & Translation	2 (8%)
Bookkeeping, Translation, & Rule	1 (4%)

Context of the Study

This dissertation study was conducted within the context of a 12-week teaching experiment designed to examine the teaching and learning of integers, specifically negative integers. The teaching experiment was comprised of a series of nine group sessions and four individual sessions. During these sessions, the students were introduced to four conceptual models for integer addition and subtraction—Bookkeeping, Counterbalance, Translation, and Relativity—through the use of various contextualized problems and activities. Although these CMIAS were introduced in the teaching experiment, it was not expected that the students would use only these models; there were opportunities for students to think about the addition and subtraction of integers freely as they engaged in activities during the group sessions. The overall structure of the teaching experiment is presented in Figure 8.

Jace Individual Context Session 1	Alice Individual Context Session 1	Kim Individual Context Session 1
Jace Individual Open Number Sentence Session 1	Alice Individual Open Number Sentence Session 1	Kim Individual Open Number Sentence Session 1
Group Session 1		
Group Session 2		
Group Session 3		
Jace Individual Context Session 2	Alice Individual Context Session 2	Kim Individual Context Session 2
Jace Individual Open Number Sentence Session 2	Alice Individual Open Number Sentence Session 2	Kim Individual Open Number Sentence Session 2
Group Session 4		
Group Session 5		
Group Session 6		
Jace Individual Context Session 3	Alice Individual Context Session 3	Kim Individual Context Session 3
Jace Individual Open Number Sentence Session 3	Alice Individual Open Number Sentence Session 3	Kim Individual Open Number Sentence Session 3
Group Session 7		
Group Session 8		
Group Session 9		
Jace Individual Context Session 4	Alice Individual Context Session 4	Kim Individual Context Session 4
Jace Individual Open Number Sentence Session 4	Alice Individual Open Number Sentence Session 4	Kim Individual Open Number Sentence Session 4

Figure 8. Structure of Teaching Experiment.

Group Sessions

The Group Sessions were guided by the mathematics that the students discussed and the misconceptions they held. I served as the teacher-researcher for this teaching experiment. A second researcher was the witness for most of the Group Sessions. He took field notes during the Group Sessions. In addition to taking field notes, he also periodically asked questions of the participants during the sessions. After each Group Session with the students, the witness and I de-briefed about the student thinking and learning that appeared to be emerging and challenging during the sessions. I conferred with him, not only about the student thinking and learning, but also considered his observations and suggestions for the next instructional moves, based on the students' responses in that session. We discussed plans for the next Group Session. After each Individual and Group Session I wrote reflections about what I noticed as the teacher-researcher, what I thought the next instructional moves should be, and why I thought that move should be made. Table 12 illustrates the outline of the CMIAS promoted during the Group Sessions, alongside the operations that were being promoted by contexts during the sessions as well.

Table 12

Structure of Group Sessions

Group Session	Conceptual Model Promoted	Operation(s)
1	Bookkeeping	Addition
2	Bookkeeping	Addition
3	Bookkeeping/Counterbalance	Addition
4	Counterbalance	Addition & Subtraction
5	Counterbalance	Subtraction
6	Translation/Relativity	Addition & Subtraction
7	Translation/Relativity	Addition & Subtraction
8	Translation/Relativity	Subtraction
9	Relativity	Addition & Subtraction

The Group Sessions began with using contexts that promoted Bookkeeping because it was used most frequently in the pilot study. The order of the other CMIAS promoted by contexts in the Group Sessions were selected based on both the pilot study and written pre-assessments from the students. After Group Sessions promoted Bookkeeping with various contexts, the Group Sessions transitioned into Group Sessions that used contexts that may have promoted Counterbalance. This was decided because both Bookkeeping and Counterbalance appeared to utilize quantities in a way that can be related (see, e.g., Liebeck, 1990 or Stephan & Akuyz, 2012). Also, the students did not appear to use Counterbalance in their written pre-assessments.

Transitioning from something that the students appeared to be familiar with to something less familiar (i.e., Bookkeeping to Counterbalance), which are related seemed like a good opportunity for learning. The next Group Sessions used contexts that promoted Translation and then contexts that promoted Relativity. Translation and Relativity also seem to be related, so contexts that promoted both of these in a Group Session were paired as transition session. The Group Sessions focused on addition and subtraction, beginning with addition of integers only and then working on both addition and subtraction of integers together.

Table 13

Integer Addition and Subtraction Open Number Sentence Problem Types involving Negative Integers

Description of Problem Type	Open Number Sentence
Addition (one positive integer given, one negative integer given) Case 1: a, b > 0 and a > b Case 2: a, b > 0 and a < b	$-a + b = \square$ $a + -b = \square$ $-a + \square = b$

Table Continues

Description of Problem Type	Open Number Sentence
Case 2: a, b > 0 and a < b	a + $\square$ = -b
	$\square$ + -a = b
	$\square$ + a = -b
Addition (two negative integers given)	-a + -b = $\square$
Case 1: a, b > 0 and a > b	-a + $\square$ = -b
Case 2: a, b > 0 and a < b	$\square$ + -a = -b
Subtraction (only positive integers given)	b – a = $\square$
a, b > 0, a > b	b – $\square$ = a
Subtraction (one positive integer given, one negative integer given)	a – -b = $\square$
	-a – b = $\square$
Case 1: a, b > 0 and a > b	a – $\square$ = -b
Case 2: a, b > 0 and a < b	-a – $\square$ = b
	$\square$ – -a = b
	$\square$ – a = -b
Subtraction (two negative integers given)	-a – -b = $\square$
Case 1: a, b > 0 and a > b	-a – $\square$ = -b
Case 2: a, b > 0 and b > a	$\square$ – -a = -b

The main focus on the design and implementation of the Group Sessions was on promoting different CMIAS with different contexts, rather than the operations or problem types alone. All of the different problem types of integer arithmetic with addition and subtraction were not included in all of these Group Sessions (see Table 13). This is because there are so many problem types and most of the sessions centered on only two or three contextual problems, with little opportunity to cover many problem types. Due to time constraints and the amount of different problem types for integer addition and subtraction, it was not possible to address all of the different problem types for addition and subtraction of integers during the Group Sessions. More specifics about the design and implementation of each of the Group Sessions are presented in Appendix D.

Individual Sessions

The teaching experiment included Individual Sessions before and after every three successive Group Sessions (see Figure 8). The Individual Sessions were implemented in pairs, with a "context" Individual Session first and an "open number sentence" Individual Session second. The Individual Context Sessions were structured to ask students to tell stories for number sentences and to respond to contextual questions. The Individual Open Number Sentence Sessions consisted of asking students to solve open number sentences for integer addition and subtraction.

Table 14

Structure of Individual Sessions

Individual Session	Content of Individual Session
Individual Sessions 1 Context	Solve and pose stories for problem with positive integers only Pose stories Solve word problems Word problems promoting Bookkeeping, Counterbalance, Translation, and Relativity

Table Continues

Individual Session	Content of Individual Session
Open Number Sentence	Solve open number sentences
Individual Sessions 2 Context	Modify positive integer story Pose stories Solve word problems Word problems promoting Bookkeeping, Counterbalance, Translation, and Relativity
Open Number Sentence	Solve open number sentences
Individual Sessions 3 Context	Modify positive integer story Pose stories Solve word problems Word problems promoting Bookkeeping, Counterbalance, Translation, and Relativity
Open Number Sentence	Solve open number sentences
Individual Sessions 4 Context	Pose stories Solve word problems Word problems promoting Bookkeeping, Counterbalance, Translation, and Relativity
Open Number Sentence	Solve open number sentences

There were two Individual Sessions conducted with Jace, Alice, and Kim after every three Group Sessions. Table 14 shows the general structure of each of the Individual Sessions. Because data for the study reported in this dissertation were drawn from these individual sessions, details about each session are elaborated upon below.

Individual Sessions 1. Jace, Alice, and Kim participated in two Individual Sessions each (See Appendix E & F) after the written pre-assessments and before proceeding to the initial Individual Session. The initial Individual Sessions had a similar structure to all of the Individual Sessions in this study; however, the initial Individual Sessions were significantly different because these sessions also incorporated questions with only positive numbers. The questions with only positive numbers were included for two reasons. First, it was important to establish comfort with each other, especially with the presence of video cameras. Beginning with problems that Jace, Alice, and Kim were possibly familiar with would help provide opportunity to establish rapport with me, as well as, possibly build some of their confidence. Second, the questions with only positive numbers were included to also gain perspective on their reasoning and misconceptions that might possibly influence the study and their understanding of negative number. This Individual Session marks the first time the students participated in session with negative integers. In the context Individual Session the students told stories for open number sentences with integers and solved contextual problems. For the contextual problems, the students were asked to solve the problem and write a number sentence. Then, the students were shown cards that had number sentences with integers and they were asked if they thought they matched or not. The Individual Number Sentence Session included students solving 20 open number sentences. There were 20 different problem types from Table 13 included.

Individual Sessions 2. Jace, Alice, and Kim participated in Individual Sessions 2 after concluding three Group Sessions that promoted Bookkeeping or Counterbalance. The Individual Context Sessions differed from the first set of Individual Sessions because they did not include problems with only positive integers. The problems with positive integers only in the Individual Context Sessions 1 were replaced with a multi-part problem that was similar to a problem presented in Group Session 2, where the students modified a story about an addition problem with positive numbers (see Appendix G). Similar to Individual Context Session 1, these Individual Sessions also included at least three number sentences with negative integers for the students to tell stories about. After the students told a story they were asked to explain why they thought it matched. After the students were asked to tell another story. I would move on to the next problem if the next story appeared to illustrate a similar conceptual model, or if the next story was exhausting the student, or due time restraints. Also similar to Individual Context Sessions 1, these Individual Sessions included problems where the students were asked to solve contextual problems and write a number sentence that they thought matched the situation. After the students solved it and wrote a number sentence, the students were encouraged to come up with another number sentence that also matched the situation. Then, the students were again shown cards with both positive and negative integer number sentences. The students were asked to explain why they thought the number sentences on the cards matched or did not match. The Individual Open Number Sentence Session 2 included more open number sentences than in Individual Open Number Sentence Session 1. Individual Open Number Sentence Session 2 included 23 problem types from Table 13 (see Appendix H). This was due to time constraints and that the students were becoming more efficient in their problem solving and able to solve more during the allotted time for meeting.

Individual Sessions 3. The Individual Sessions 3, which was a context session and then open number sentence session, followed three Group Sessions on where Counterbalance and Translation promoted through contexts. These individual context sessions, Individual Context Sessions 3, were similar to Individual Context Sessions 2 (see Appendix I). All of the same problem types were included, with the exception that one more part was added to the problem where students modified a story about an integer addition problem. The modification came with the inclusion of both number types, -10 – -4 and 10 – -4. This was done to give students more time for consideration of subtracting a negative and also to examine the subtracting of a negative number further. Again, more problem types for the Individual Open Number Sentence Sessions were covered (see Appendix J). This was again due to time constraints and the efficiency of the students solving the problems.

Individual Sessions 4. The Individual Sessions 4, which was a context session and open number sentence session, followed three Group Sessions that promoted Translation and Relativity. The Individual Context Sessions 4 were similarly structured to the Individual Context Sessions 2 and 3. In fact, some of the exact same problems from Individual Context Session 1 were included in Individual Context Session 4. Since these Individual Sessions occurred twelve weeks later, the students more than likely did not remember those problems or how they had solved them twelve weeks ago. Also, using the exact same problems in Individual Sessions 1 and 4 provided a perspective of the students solving the same problem weeks later. Students again posed stories for number sentences with integer addition and subtraction in Individual Context Sessions 4 (see Appendix K). Although the same problem types were used, one more problem type was added (i.e., 1 – -2 = □). This problem was added to provide another opportunity into looking at how the students conceptualized subtracting a negative number. Another difference from Individual Context Sessions 2 and 3 is that the problem where students modified a story based on an addition problem with positive integers was removed from Individual Context Session 4. For this problem, most students posed a story involving discrete items for the addition problem with positive integers only. Therefore, when students would modify this story about positive integers their stories with negative integers they would end up illustrating Bookkeeping thinking. The decision to remove this was made to allow for the students to have the freedom to use any CMIAS they wanted to pose stories for open number sentences with integers, while not being influenced by other problems during the session. A problem was also added to this context session that was not included in the previous interviews. The problem added was a problem involved finding the distance between two integers with temperature. This type of problem came out in Group Session 8. Including this problem on in Individual Context Session 4 provided the opportunity for perspective into the students' learning from Group Session 8 to Individual Context Session 4. This was the problem added to this Individual Session:

> It is -4° Fahrenheit in Siberia in the morning. It is -7° Fahrenheit in Siberia in the afternoon. What's the difference in temperatures?

The Individual Open Number Sentence session included the same open number sentences that were in Individual Open Number Sentence Session 1, but included more problems since the students were more efficient. Individual Open Number Sentence Session 4 included 25 open number sentences (see Appendix L).

The common problems across all four Individual Context Sessions were telling stories for open number sentences with integer addition and subtraction and solving contextual problems that promoted four conceptual models (i.e., Bookkeeping, Counterbalance, Translation, Relativity). The Individual Context Sessions 2, 3, and 4 included a problem that was considered to promote both Translation and Relativity. The Individual Context Sessions 2 and 3 both included a problem where students needed to modify a number sentence. These commonalities across the individual context session are shown in Figure 9.

	Individual Context Session 1	**Individual Context Session 2**	**Individual Context Session 3**	**Individual Context Session 4**
Problems where students were asked to pose stories.	-17 + 12 = □ -2 – 3 = □ -5 + □ = -7	-20 + 13 = □ -4 – 7 = □ -5 + □ = -7	-15 + 4 = □ -3 – 5 = □ -1 + □ = -2	-12 + 7 = □ -2 – 10 = □ -7 + -2 = □
Problems where students modified a story.		Tell me a story for 9 + 6. How would your story change if we changed the number sentence from: 9 + 6 to -9 + 6? 9 + 6 to -9 + -6? 9 + 6 to 9 + -6? 9 + 6 to -9 – -6?	Tell me a story for 10 + 4. How would your story change if we changed the number sentence from: 10 + 4 to -10 + 4? 10 + 4 to -10 + -4? 10 + 4 to 10 + -4? 10 + 4 to -10 – -4?	
Bookkeeping Contextual Problems	Edward lost 12 pencils in his messy office. While he was cleaning his office he found 5 pencils. How many pencils are still lost in his messy office?	Raquel needs twenty-two cupcakes for her party. She only baked twelve cupcakes. What's Raquel's situation?	Edward lost 12 pencils in his messy office. While he was cleaning his office he found 5 pencils. How many pencils are still lost in his messy office?	Jose needs eighteen pizzas for his party. He only bought seven pizzas. What's Jose's situation?
Counterbalance Contextual Problems	You had 5 sad days and 2 happy days one week. What kind of week did you have?	Josiah did twenty-two good deeds and eight bad deeds at school this week. What kind of week did Josiah have?	Sophie did nine bad deeds and six good deeds at school this week. What kind of week did Sophie have	You had 4 sad days and 3 happy days one week. What kind of week did you have?
Translation Contextual Problems	It is 2° Fahrenheit outside. The wind chill makes it feel 10° colder. What temperature does it feel like?	It was 8° Fahrenheit outside. The temperature dropped 20° Fahrenheit. What is the temperature now?	It was 6° Fahrenheit outside. The temperature dropped 11° Fahrenheit. What is the temperature now?	It was 4° Fahrenheit outside. The temperature dropped 12° Fahrenheit. What is the temperature now?
Relativity Contextual Problems	Kim and Jordan were playing an online game where points earned are displayed online. Kim felt competitive and noticed that in the first game she was 2 points behind Jordan. For the second game, Kim lost to Jordan by 3 points. How does Kim start the third game?	The Red Birds and Dragons were playing against each other in a baseball game. The Red Birds were down 4 runs in the first inning and gained 6 runs against the Dragons in the second inning. How did the Red Birds start the third inning?	Emma and Kirsten were playing an online game where points earned are displayed online. Emma felt competitive and noticed that in the first game she was 4 points behind Kirsten. In the second game, Emma lost to Kirsten by 7 points. How does Emma start the third game?	The Red Birds and Dragons were playing against each other in a baseball game. The Red Birds were down 10 runs in the first inning and gained 12 runs against the Dragons in the second inning. How did the Red Birds start the third inning?
Contextual Translation/Relativity Story		Angela and her dad decided to plant a tree in their backyard. Angela found a spot in the yard that she wanted to plant the tree. Her dad told her to move the tree from that spot to the right 10 feet. Angela moved the tree to the position that her dad wanted, but then she decided to move it 12 feet to the left. Where did Angela eventually end up planting the tree?	Angela and her dad decided to plant a tree in their backyard. Angela found a spot in the yard that she wanted to plant the tree. Her dad told her to move the tree from that spot to the right 6 feet. Angela moved the tree to the position that her dad wanted, but then she decided to move it 18 feet to the left. Where did Angela eventually end up planting the tree?	Angela and her dad decided to plant a tree in their backyard. Angela found a spot in the yard that she wanted to plant the tree. Her dad told her to move the tree from that spot to the right 12 feet. Angela moved the tree to the position that her dad wanted, but then she decided to move it 20 feet to the left. Where did Angela eventually end up planting the tree?

Figure 9. Common Problems Across Individual Context Sessions.

The common problems across the all four Individual Open Number Sentence Sessions were at least 20 open number sentences with integer addition and subtraction. The common problems were not the exact same problems across the interviews, with the same numbers used. Rather, the problems were the same problem types (see Table 13). The first and last Individual Open Number Sentence Sessions used the same problems types with the same numbers. For a comparison of the open number sentences used across the individual number sentence sessions see Figure 10. The quantity of open number differed between the sessions because of time restraints for the given session and how efficient Jace, Alice, and Kim were at solving the problems. Each of these problems were presented to the students on an individual sheet of paper. The students only had markers as tools to solve these problems and a sheet of paper with the open number sentence provided on it.

Individual Open Number Sentence Session 1	**Individual Open Number Sentence Session 2**	**Individual Open Number Sentence Session 3**	**Individual Open Number Sentence Session 4**
-20 + 15 = □	-16 + 4 = □	-18 + 12 = □	-20 + 15 = □
12 + -16 = □	20 + -33 = □	15 + -24 = □	12 + -16 = □
-4 + □ = 10	-6 + □ = 15	-3 + □ = 14	-4 + □ = 10
-7 + □ = -2	-6 + □ = -1	-9 + □ = -3	-7 + □ = -2
□ + -3 = 7	□ + -2 = 17	□ + -4 = 13	□ + -3 = 7
□ + 13 = -5	□ + 19 = -4	□ + 25 = -2	□ + 13 = -5
-8 + -7 = □	-12 + -5 = □	-17 + -6 = □	-8 + -7 = □
-2 + □ = -10	-4 + □ = -19	-5 + □ = -21	-2 + □ = -10
□ + -9 = -16	□ + -9 = -21	□ + -9 = -17	□ + -9 = -16
10 – 12 = □	5 – 9 = □	12 – 18 = □	10 – 12 = □
1 – □ = 3	4 – □ = 6	3 – □ = 4	1 – □ = 3
-5 – 4 = □	-9 – 8 = □	-5 – 3 = □	-5 – 4 = □
2 – -3 = □	3 – -4 = □	1 – -3 = □	2 – -3 = □
-1 – □ = 8	-2 – □ = 9	-2 – □ = 10	-1 – □ = 8
2 – □ = -10	6 – □ = -10	4 – □ = -12	2 – □ = -10
□ – -1 = 6	□ – -1 = 4	□ – -2 = 5	□ – -1 = 6
□ – 8 = -5	□ – 9 = -3	□ – 6 = -2	□ – 8 = -5
-15 – -4 = □	-11 – -2 = □	-12 – -4 = □	-15 – -4 = □
-12 – □ = -13	-15 – □ = -16	-10 – □ = -11	-12 – □ = -13
	□ – -3 = 2	□ – -3 = 1	□ – -2 = 1
	□ – -4 = 0	□ – -5 = 0	□ – -3 = 0
	12 + □ = 8	15 + □ = 9	17 + □ = 8
		8 + □ = -5	6 + □ = -2
		□ + 2 = 0	□ + 4 = 0
		-4 – 10 = □	-2 – 8 = □

Figure 10. Open Number Sentences given in Individual Open Number Sentence Sessions.

Data Sources

A variety of data was collected to examine students' thinking and the students' learning experiences throughout the teaching experiment. However, for the purposes of the study reported in this dissertation, the main source of data came from the Individual Sessions and included video recordings of all sessions, transcripts, students' drawings, and a teacher-researcher journal. Collecting data from a variety of sources aids in the trustworthiness of qualitative research.

The commognition framework (Sfard, 2008) points to components of discourse that are outside of the language itself. For example, some of the students' communication occurred through their drawings. Video recordings allow for these types of data to be collected (e.g., drawings, written number sentences), thus, all sessions were recorded to capture the discourse between each of the students and myself. At least two cameras were used in each session to capture students' gestures, writing, and drawings. Having two cameras also as a backup was a good decision because occasionally there were technical difficulties and a couple sessions only have one of the camera's footage. All of the students' drawings were collected and examined. I wrote entries in a teacher-researcher journal after each session and again after watching the video footage of all of the sessions.

Transcripts

All Individual Sessions were transcribed verbatim since word use is an important component to the commognitive framework. Because the commognitve framework is more than just word use, the distinctive gestures that students made or the drawings that they produced were also transcribed along with the word use. The gestures and drawings were fully transcribed and, if possible, when the gesture or drawing was paired with verbal it was transcribed next to the verbal word use in parentheses. The gestures were transcribed by providing a description of the hand or finger movements either in the air or towards the student's paper. Although everything was transcribed, the verbal contributions of the students constituted the word use of the commognitive framework for this study.

Students' Drawings

All of the written work, or drawings, from the students was collected and analyzed alongside the video recordings and transcripts. The written work included any form of writing that the students created. For example, the students produced drawings (i.e., numbers lines or tallies when they solved open number sentences. These types of drawings are important to understanding the student's thinking paired with their word use. Drawings or writing of symbolic notations are considered to be the visual mediators of the commognitive framework for this study.

Teacher-Researcher Journal

At the end of each of the Individual Sessions, I wrote in a journal about my interpretation of the events that occurred. As I wrote, I reflected on what the students were thinking or learning. After I transcribed each session, I re-read the journal entry and sometimes added more components to the journal. In addition to adding more descriptions to the journal, I often included excerpts of the transcript to support some of the statements made in the journal. Journaling was a way to record the development of the tasks of the teaching experiments, which were described earlier in this chapter.

Journaling and transcribing was a tool to start noticing themes that were present in the data.

Data Analysis

This study, which was guided by the theoretical lens of secondary intuitions (Fischbein, 1987) and commognitve theory (Sfard, 2008), took a different analytical approach than typically utilized with commognition. That is, Sfard (2008) in addition to presenting commognition theory, discusses what discursive analysis could be like. Suggested analysis capitalizes on discursive techniques like “realization trees” and identification of “objects” within discourse. In fact, our scholarly community recognizes Sfard’s contribution of commognition is considered a “strong example” of a guiding theoretical lens because researchers from other fields have utilized it (Jablonka, Wagner, & Walshaw, 2013).

Although the discursive techniques of analysis that Sfard (2008) suggests are useful for unpacking the richness of mathematical discourse aligned with commognitive theory, I found these techniques cumbersome when trying to make sense of the Individual Session data. Other discourse analysis experts, like Gee (2011), stated, “In my view, no one theory is universally right or universally applicable. Each theory offers tools which work better for some kinds of data than they do for others” (p. iv). Gee pointed to the need for discourse analysis to often be conducted on smaller amounts of data, like a singular lesson, 10 minutes of excerpts. The discursive approach conducted by others that have used commognition may entail coordinating up to 110 corpuscular definitions to your data, to help ensure scientific authenticity to the qualitative work. Also, much of the work is focused primarily on coordinating word use. With a smaller amount of data and different research question, this may have worked well. Similarly, I wanted to focus on the visual mediators as much as the word use.

Because the recommended tools did not aid in a handling of the data to address my research question, I knew I needed a different analytic approach. When Sfard was questioned about my interpretation of her tenets of commognition and the type of analysis I hoped to conduct, Sfard replied:

> While it is true that I often don't recognize my ideas in what other people say about them, or in the way they use them, I also eschew protesting, or even trying to ‘correct.’ A thought once released in a publication has life of its own, and it is no longer up to the original author to try to keep her original ownership. So, worry not: I am not here policing the use of the discursive approach. You should work with it according to your understanding and only if you can feel that it is really helpful to you (Sfard, personal communication, February 27, 2014).

Once I felt freedom in my use and interpretation of commognitive theory, I identified the most useful aspects of commognitive theory for my study. These included: (a) the definition of learning as a change in discourse and (b) the utilization of the tenets of mathematical discourse of commognition to parse and understand my data (i.e., word use-transcripts of verbal interactions, visual mediators- drawings and writings of students, narratives—CMIAS demonstrated by students, routine- patterns in the discourse) as well as define mathematical thinking. Because I wished to refine the initial descriptions of CMIAS and describe student use of these, constant comparative methods (Merriam, 1998) were selected. And, the method to describe learning drew upon the definition of learning from Sfard and identifying changes in those discursive tenets.

Transcripts of the verbal contributions of the students were considered the data source to find their word use. The visual mediators entailed anything that students drew or wrote during a session and these were scanned in. The act of investigating the CMIAS, through the use of both transcripts and drawings, was considered to be an act of describing the students' narratives. Looking for repetitive patterns in word use, visual mediators, and narratives was considered examining the routines. With commognition, all of these tenets describe the students' thinking and each of the tenets represent thinking (Sfard, 2008). There is no longer the Descartes dualism of what is "in the head" and what is "reality." Rather, it is all viewed as mathematical thinking—the words spoken, drawings produced, CMIAS used, and repetitive patterns among these. The tenets of commognitive theory provided a way to begin looking at the data. The tenets provided guidance on how to begin to make sense of the massive amount of qualitative data.

Analysis of Individual Session Data Across All Students

I analyzed data from the Individual Sessions to examine the ways students used conceptual models (e.g., counterbalance, bookkeeping, relativity, transformation, rule) as they attempted to make sense of integers. All three of the students' responses to solving open number sentences and posing stories were used. This provided the opportunity to examine what CMIASs students were using, while not being constrained to the initial CMIAS descriptions since all of the contextual problems had been developed with the initial CMIAS descriptions. Data analysis occurred in four phases, which are described below.

Phase 1. The research question was addressed by analyzing video recordings, transcripts, and drawings produced for solving open number sentences and posing stories during the Individual Sessions for all three students. First, I transcribed all of the video recordings of the Individual Sessions (i.e., 3 Students x 2 Individual Session Types x 4 Sessions = 24 sets transcripts) for all three of the students. As part of the transcription process, I wrote in a journal and noted emergent themes. See Appendix M for an excerpt of the teacher-researcher journal. For example, I reflected on how the students used empty number lines or drew tallies to solve the problems. These reflections and noticings varied as I transcribed. For example, I reflected on how the students seemed to transition from using number lines to solve a problem to using number line as a way to explain or justify their answer after solving it and I reflected on how I thought the CMIAS were present in the situation. Also, as I transcribed, I not only transcribed the word use, but I also transcribed how the students were drawing or writing number sentence. This was because often the students' words were paired with their drawings or number sentences they were writing.

Second, I watched and re-watched the videos, read and re-read both the transcripts and written student work from the individual interviews for emergent themes by looking at and considering word use, visual mediators (e.g., drawings, number sentences), narratives (e.g., ways that students handled the negatives), and routines (e.g., repetitive actions) established. Some of the emergent themes aligned well with the initial descriptions of the CMIAS. For example, Kim often shared about "going deeper in the negatives" and this aligned well with Translation. However, Alice, Kim, and Jace each made comparisons from one number sentence to a different numbers (e.g., comparing -2 + -3 to 2 + 3) and this theme was not present in the current CMIAS descriptions. Similarly, Alice, Kim, and Jace each referenced use of the "commutative property" at some point in the teaching experiment and this use was not present in the initial CMIAS descriptions. After time was spent transcribing, journaling, examining the data, and reflecting on these themes, I decided to address the refinement of the CMIAS and what CMIAS the students may have used first needed to be informed by their word use (i.e., transcripts of their word use) and visual mediators (i.e., drawings produced by the students).

Third, after the transcription and reflection process noted above, the unit of analysis was determined to be each of the students' responses to solving an open number sentence or posing a story during the Individual Sessions. The data that captured the students' verbal and written responses best was the transcripts and the drawings produced. Drawings, the visual mediators, were considered to be anything that the students' produced on paper. Thus, a number line, tallies, and number sentences were all considered drawings. The transcript, incorporating students' word use and visual mediators, for each problem and the drawings produced alongside it were examined as a pair and considered the unit of analysis. For example, a student's entire response (i.e., the transcript of verbal response and anything drawn on paper) when solving a specific open number sentence, like $-20 + 15 = \square$, during the Individual Open Number Sentence Sessions 1-4 was considered a unit of analysis. Similarly, both the verbal and written products of a posing a story or a solving a contextualized problem in the Individual Context Sessions 1-4 was the other unit of analysis.

The main accomplishment of Phase 1 was the determination of the unit of analysis, which was the students' entire word use (captured by transcript) and visual mediators produced (captured by both transcripts and drawings produced by students) for posing stories and solving open number sentences both the Individual Context Sessions 1–4 and Individual Open Number Sessions 1–4, respectively. For Alice, this meant that there were 93 units of data from the Individual Open Number Sentence Sessions and 23 units of data from the Individual Context Sessions. For Jace, this meant that there were 93 units of data from the Individual Open Number Sentence Sessions and 28 units of data from the Individual Context Sessions. For Kim, this meant that there were 93 units of data from the Individual Open Number Sentence Sessions and 24 units of data from the Individual Context Sessions. In total, for each of the Individual Sessions, this meant that there were 93 x 3 = 279 units of data from the Individual Open Number Sentences and there were 23 + 28 + 24 = 75 units of data from the Individual Context Sessions. Collectively, that meant that there were 354 units of data that were examined for refinement of the CMIAS and coded in Phases 2–5, which are described next. See Appendix N for an example of what a unit of data looked like.

Phase 2. Each of these units of data (i.e., the students' entire transcript and drawings produced when posing a story or solving an open number sentence) was analyzed for each of the sub-questions of the research question by using constant comparative methods (Merriam, 1998) with the CMIAS descriptions as the initial codes (see, e.g., Wessman-Enzinger & Mooney, 2014) as well as the option to code Other. The data were organized with respect to each of the discursive tenets from commognition. Each of the transcripts, which included the students' verbal use about integers (i.e., word use) and descriptions of drawings (i.e., visual mediators), were paired with the all of the drawings produced by the students (i.e., visual mediators). Each of these 354 units of data was examined by the witness and myself.

Constant comparative methods were selected as an analytic tool to refine the CMIAS because the CMIAS already had initial descriptions and an aim of this study was develop more robust descriptions of them and generate new possible CMIAS. Furthermore, the act of coding with the CMIAS was viewed as a way to describe the students' narratives (i.e., narratives) or the students' use of the CMIAS, which also addressed the second part of the research question. Constant comparative methodology allowed refinement of the CMIAS, with the allowance of generating new CMIAS to describe the students' narratives betters.

Each of these units of data was first coded individually by both the witness and me with the initial CMIAS or Other (Wessman-Enzinger & Mooney, 2014). The descriptions of the initial CMIAS, which are described in both Chapter I (see, e.g., pp. 9–13) and Chapter II, were compared to the units of analysis for solving open number sentences and posing stories for all three students. The witness and I allowed the option to code "Other" when the current descriptions of the CMIAS broke down or were not sufficient. The coding of Other was thought to highlight areas that would need to be discussed for refinement in the coding process to develop more robust descriptions of the CMIAS. Because the initial descriptions of the CMIAS were developed from contextual stories, this presented little issues in coding the units of data from the Individual Context Sessions. However, this was more challenging when coding the units of data from the Individual Open Number Sentence Sessions. For this reason, having the code of Other was important and highlighted areas to pay attention to for the refinement of the CMIAS. For example, when Jace solved $5 + \square = -3$, he responded, "I figured out that eight is greater than five, so that'd be negative…three because eight is bigger than five and eight is the negative number." This was coded as Other in the first pass of coding. Both the witness and I coded this as Other because it was agreed that Jace used magnitude comparisons here by comparing the size of 5 and 3. It was unclear in the first round of coding how magnitude was related to the initial descriptions of the CMIASs and so anything that had apparent magnitude ideas like this was coded as Other. This was an issue that was negotiated in Phase 3 and 4.

In this initial coding of the data, where the initial CMIAS descriptions and Other were utilized, Rule was the most utilized code by both the witness and I for all three of the students. In fact, Rule was coded for more than half of the units of data. For this reason, the category Rule was examined further in Phase 3 to investigate the differences in the units of data that were coded with Rule.

As described above, the witness and I met to discuss each of these initial sets of codes, negotiate differences, and note places of difficulty in coding weekly or bi-weekly from August 2014 to December 2014. We met to compare our codes and discuss discrepancies on this first pass of coding. The codes, the agreement, and notes about the disagreement were recorded in a coding sheet. An example of a Phase 2 coding sheet from Kim's Individual Open Number Sentence 1 is provided in Appendix O. In addition to discrepancies and negation, both the witness and I met to discuss challenges in using the current CMIAS descriptors to describe the students' narratives of solving the open number sentences or posing a story. For example, the witness and I discussed that coding Rule was ambiguous and there seemed to be significant differences in some of the units of analysis that were coded with Rule. Additionally, how students made many analogies to whole numbers to solve some problems came up in discussion. For a problem like -2 +

-3, students often reasoned that because two plus three is five, negative two plus negative three must be negative five. We discussed and debated whether this should be included as Rule since this analogous reasoning seemed different than other types of reasoning. Similarly, for the problem 5 + -3, student may have used a Rule that adding a negative is the same as subtracting, and solve 5 + -3 as 5 – 3. The analogous reasoning seemed different from this implicit rule that the students were using; yet, because the analogous reasoning was based on connecting it to a whole number rule, we agreed to code this as Rule still and the other was coded as Rule as well in Phase 2. For this reason, we knew we needed to do a second pass and look at other possible emergent theme within the units of data coded Rule in Phase 3.

Another challenge that emerged during this initial pass of the data was how to code strategies that used directed distances and strategies that used distance without direction. We negotiated the meaning the Translation to be any directed distance and distance without directions during this phase. More about details about this discussion is in Chapter IV. Because this first pass of the data using the CMIAS was messy and difficult, it was important to look for commonalties and challenges across all of the units of data for all of the students, do generate better CMIAS descriptions or establish new CMIAS in Phase 3.

Phase 3. All of the units of data for all three students, across all of the Individual Session interviews, were coded with the initial CMIAS or Other in Phase 2. After Phase 2, where the first pass of all of the units of data were coded this way, all of the units were organized by their codes and re-examined during Phase 3. For example, all of the units that were coded with Rule were re-examined. All of the units that were coded

Bookkeeping were re-examined. And, this was repeated for each code. Each of these units in their respective groups (e.g., Bookkeeping, Rule) were examined and compared to the initial CMIAS descriptions. For example, each of the units that were coded as Translation were compared to initial definition of Translation. When there were parts of unit of data that did not adequately align with the definition, like counting, it was noted. Then, if there multiple units of data that also contained that attribute that were not present in the initial CMIAS definition, the definition was modified. As I modified the previously defined descriptions of the CMIAS, I met with the witness several times throughout this process to discuss these modifications from December 2014 to January 2015. A main discussion during these meetings was focused on that Rule was coded too much in Phase 2 and contained different ways of reasoning. The units of analysis for Rule included reasoning that focused on the use of the minus sign, reasoning that drew upon analogies, and reasoning that used mathematical properties like the commutative property. The witness and I created new CMIAS from Rule through discussion and negotiation.

Phase 4. Last, the witness and I then coded each of the units of data with the new descriptors, which are described as the Refined CMIAS in Chapter IV, in a second pass of the data. The teacher-research and witness met weekly or biweekly from January 2015 to March 2015. They met, compared codes, and negotiated any differences for all of the data for all three students. The codes, negotiated codes, and notes were recorded in a coding sheet. An example of a Phase 4 coding sheet from Kim's Individual Open Number Sentence Session 1 is provided in Appendix P. Through this repetitive process of coding, comparison, discussion, and negotiation the new descriptions of the CMIAS were further modified in this last phase. With consistent weekly or bi-weekly meetings of both the witness and myself this process of coding also helped to further refine the models. For example, the process of coding highlighted the need in Phase 4 helped to clarify Bookkeeping as a change to a singular quantity, to distinguish it from Counterbalance and some of the data was re-coded to account for this modification. Also, during Phase 4, Analogy was modified further to include two-subgroups. Descriptions of the refined CMIAS and decisions made during this refinement in Phase 4 are described in Chapter IV.

Phase 5. Phase 4 included the coding, modifying, and re-coding of data. Once Phase 4 was completed, the codes of the refined CMIAS for each unit of data were interpreted as the students' narratives for solving the open number sentence in Phase 5. Some of these counts and percentages calculated of the students' uses that were generated during Phase 5 are also provided in Chapter IV. These are provided as a way to make sense of the qualitative data to highlight the students' use of the CMIAS. The percentages were also utilized to help illustrate the differences in CMIAS use that different students use, as well as, the use in different types of problems: contextual and symbolic. Then, the percentages of use for each CMIAS across each of the Individual Sessions for each student were examined, to make sense of the students' use of the CMIAS. Although all the percentages for all three students are provided in Chapter IV, Alice was selected as a student to provide further insight into what these uses look like. A qualitative account with examples of her work is provided for a focused and in-depth account of her use of the CMIAS.

Analysis of Individual Session Data for Jace

Capturing development, or learning, as a change in discourse (Sfard, 2008) over a significant amount of time requires a way to productively manage and reduce the data in order to make sense of the data to describe learning. To address the second part of the research question, about the ways students use conceptual models of the integers as they learn about integer addition and subtraction, I analyzed the changes exhibited in Jace's thinking across the Individual Sessions. I selected Jace because he had attended the most sessions throughout the teaching experiment, missing only one of the Group Sessions. Also, Jace was the only student in the study that eventually used all of the CMIAS at some point in the study (see Chapter IV).

Drawing upon Sfards' (2008) definition of learning as a *change in discourse*, I analyzed the data with regard to changes in word use, visual mediators, narratives, and routines. Although learning is not defined as an "acquisition" of getting the correct answer, the identification of correct and incorrect answers helped narrow the data. For example, as Jace solved open number sentences during his Individual Open Number Sentence Sessions he solved some problems correctly across the four sessions. Other open number sentences, he solved incorrectly across the four sessions. And, other open number sentences he solved incorrect across the sessions and eventually got them correct. A single open number sentence type (see Table 13) from each of these categories was selected for examination. Three open number sentence types were selected: $-a + \square = b$ ($a, b > 0$ and $b > a$), $-a – b = \square$ ($a, b > 0$ and $a > b$), and $-a – \square = -b$ ($a, b > 0$ and $b > a$). I identified and described Jace's word use, visual mediators, narratives, and routines for each of these three open number sentences. That is, each of these tenets across the four sessions were placed in a chart next to each other. I then examined and described the changes in each of the tenets by looking across the sessions for similarities and differences for each tenet of commognition (i.e., word use, visual mediators, narratives, routines) for each of the open number sentences. Descriptions of these changes are provided, by each commognitive tenet, in Chapter V.

Table 15 below highlights how each of the two parts of the research question is related to the data sources used and the data analysis.

Table 15

Summary of Research Question, Data Sources, and Data Analysis

Research Question	Data Sources	Data Analysis
In what ways do fifth-grade students use conceptual models of the integers (e.g., Bookkeeping, Counterbalance, Translation, Relativity, Rule) as they		
(a) attempt to make sense of the integers	Transcripts Drawings	Word use and visual mediators were used to determine the students' narratives by examining routines, or repetition of thinking, with constant comparative methods (Merriam, 1998).
And (b) learn about integer addition and subtraction?	Transcripts Drawings Teacher-researcher journal	Identifying changes in word use, visual mediators, narratives, and routines with a qualitative description of changes of over time of Jace solving three different open number sentences.

CHAPTER IV

REFINEMENT OF THE CONCEPTUAL MODELS FOR INTEGER ADDITION AND SUBTRACTION & THE STUDENTS' USE OF THE REFINED CONCEPTUAL MODELS

This chapter begins with describing the refined Conceptual Models for Integer Addition and Subtraction (CMIAS) and how they were refined. Descriptions of the refined CMIAS are purposefully mathematical and focus on the roles of the integers within each of the conceptual models. The decisions that were made how the models were refined are provided with the supporting data. After the refined models are described, this chapter transitions to the ways that Alice, Jace, and Kim utilized the CMIAS in the Individual Sessions. The purpose of the descriptions of these uses of the three students is to provide broad descriptions of how the three Grade 5 students used the CMIAS. Then, this chapter concludes with an in depth illustration of how one student, Alice, used the CMIAS throughout the Individual Sessions of the teaching experiment.

Refinement of CMIAS

The CMIAS are ways of reasoning, thinking about, and mathematical uses of adding and subtracting the integers both symbolically and within contexts. The original CMIAS were Bookkeeping, Counterbalance, Translation, Relativity, and Rule. After data analysis, the CMIASs have been modified to seven different conceptual models:

Bookkeeping, Counterbalance, Translation, Relativity, Analogies, Algebraic Reasoning, and Proceduralization. Each of these seven modified CMIASs and how they were refined is described next, with the coded data from Phase 2 being reported on as well as the refinements and refinement decisions of Phases 3 and 4. For all the CMIAS reported on, the emphasis is that these are ways of thinking and using the integers and being right or wrong is not important.

Each of the descriptions of the CMIAS were first modified by examining the units of data coded in Phase 2 from Alice, Kim, and Jace from the Individual Open Number Sentence Sessions and the Individual Context Sessions. However, the units of data from the Individual Context Sessions did not provide insight into the refinement of these models. That was to be expected given that the initial CMIAS descriptions were initially developed from examining the stories that student generated when they posed stories for integer open number sentences. After those modifications were made using the units of data from the Individual Open Number Sentence Sessions, the descriptions were further refined to include clear and consistent descriptions that defined role of zero and the role of the integers in addition and subtraction. This was done from my mathematical perspective, rather than the units of data. The mathematical descriptions of the CMIAS as well as this process of the refinement are provided next.

Bookkeeping

Approximately 13% of the students' responses to the Individual Open Number Sentences (each coded unit of data) exhibited use of the Bookkeeping model of thinking. However, all of these units of data came from Alice. Of Alice's 93 units of data from the Individual Open Number Sentence Sessions, there were 36 units coded as Bookkeeping.

Although only Alice's units were used in this refinement, the original description did not fully capture the mathematical use in the students' responses. Based on my analysis of the data, I made one modification to the definition (see Table 16). That change involved the clarification of wording to help distinguish it from other models.

Table 16

Original and Refined Bookkeeping CMIAS

Original CMIAS Description from Wessman-Enzinger & Mooney (2014)	Refined CMIAS Description
A conceptual model of *Bookkeeping* model involves using integers to describe gains and losses. The use of zero in this model represents a status of neither loss nor gain. The bookkeeping model can represent a gain and loss of any item and is not necessarily limited to the context of money. For example, gains and losses can be conceptualized through such scenarios as owing and acquiring candy bars or wanting and receiving baseball cards. (p. 203)	*Bookkeeping* is utilized when the integers are used in a way to describe losses and gains of quantities. The zero in the bookkeeping conceptual model represents having neither a specific gain nor loss. In Bookkeeping, the positive and negative integers represent a gain or loss of something applied to a singular quantity. That is, the addition of a positive integer may be treated as gain to the singular quantity and the addition of a negative integer may be treated as a loss to the singular quantity. Or, the subtraction of a positive integer may be treated as a loss to the singular quantity and the subtraction of a negative integer may be treated as a gain to the singular quantity. Bookkeeping points to thinking that is mostly quantitative reasoning without explicit reference to order and without reference neutralization. Although the positive integers are treated as an increase in the quantity and the negative integers are treated as loss in quantities, the addition of positive integers to a negative quantity may also be considered a loss of negative quantity. A distinguishing element of Bookkeeping compared to other models is that the gain or loss is being applied to a singular quantity.

Clarification of wording. The clarification of wording included defining integer addition and subtraction as a gain or loss to a *singular quantity*. The refinement of recognizing gains and losses applied to a singular quantity was made from all of Alice's responses. All of her responses (the 36 units of data) utilized discrete quantities without order and without neutralization. All of the responses included a gain, adding discrete quantities to an already existing quantities, or removing quantities from an existing quantity. For example, when Alice solved -18 + 12 = □, she stated:

> I did eighteen lines with (points at the green tallies with finger), which that was the eighteen negative. I crossed off twelve (points at the right with the pink). And, I still had six negative left (points at the tallies that were on the left that had not been crossed off).

In this example, Alice drew one singular quantity (i.e., a set of tallies) and changed this singular quantity by crossing off tallies (i.e., a loss applied to the singular quantity). This type of thinking seemed distinct from Counterbalance because positive and negatives were not being compared. Also, this did not seem like Translation because there was not movement, shift, or continuous distance incorporated. Although these types of encounters happened with Alice frequently, they didn't occur for Kim or Jace. Although none of Kim's of Jace's units of data influenced the refinement of this definition, all of Alice's responses included her verbally discussing a gain or loss to a singular quantity and applying a gain or loss to that set of drawn tallies.

Counterbalance

Approximately 4% of the students' responses from the Open Number Sentence Sessions (each coded as a unit of data) was coded Counterbalance. That is, only 10 units of the 279 units of data were coded as Counterbalance. All 10 of those units were from Alice's Open Number Sentence Sessions. Based on my analysis the data, there were two refinements made to counterbalance (see Table 17).

Table 17

Original and Refined Counterbalance CMIAS

Original CMIAS Description from Wessman-Enzinger & Mooney (2014)	Refined CMIAS Description
A conceptual model of *Counterbalance* involves using positive and negative integers to balance, or cancel, each other out. This model is similar to the "Balanced Metric," cancellation, and chip modeling concepts found in the literature (e.g., Battista 1983). The zero in this counterbalance model indicates neutralization. The distinguishing element of this model is that the quantities always remain, even when neutralized. (p. 203)	*Counterbalance* is utilized when the negative and positive integers are treated as two separate quantities in a way that balances each other out. The zero in the counterbalance conceptual model represents a status of neutralization. Positive and negative numbers in Counterbalance are not just opposites, but opposites that neutralize. Addition of integers, whether positive or negative, represents joining quantities where the equal number of positive and negative integers are neutralized and the sum is the integers not neutralized. Subtraction of integers, whether positive or negative, represents removing quantities, which may entail removing quantities from a status of neutralization. Similar to Bookkeeping, Counterbalance points to thinking that is mostly quantitative. A distinguishing feature of Counterbalance from other models, and specifically Bookkeeping, is that there are two quantities that always remain present in the Counterbalance, even when neutralized. The absolute value, or magnitude, of these two different quantities may be compared to solve addition and subtraction problems.

One change involved clarifying wording the wording of the description. The other change addressed ways the students' ways of thinking about the integers that were not adequately described in the original definition. This change pertained to the mathematical use of absolve value, or absolute magnitude, comparisons.

Clarification of wording. All 10 of the units of data included a comparison of two different discrete quantities. The original definition included two quantities without explicitly stating it, when it included, "using positive and negative integers to balance, or cancel, each other out." However, within these units of data, sometimes only two positive numbers were compared, rather than a positive and negative number. For example, when Alice solved 12 – 18 = □, she drew 12 tallies above 18 tallies. She compared these two quantities, and determined the answer was -6:

> Well, I did twelve (points at the green tallies.) as the twelve (points at 12 in the number sentence). And then, I did eighteen (points at the pink tallies). And this is twelve (points at the left side of the pink tallies). So I (takes hand and covers green and pink tallies) knew that the six extra ones (still covering both sets of twelve tallies, uses right hand to point at the uncovered pink tallies) were the answer.

Alice, using her hand to cover up the tallies, illustrated implicit neutralization. Alice used Counterbalance, but she did not do with positive and negative numbers. Alice used Counterbalance with two quantities, even when both were positive, like 12 and 18. Also, within these 10 units of data, there was a unit of data that was accidentally coded as Bookkeeping. For this reason, there was a need for clarity in wording. The wording that a comparison of “two quantities” was added to account for the use of two quantities and to help provide clarity in the distinction from Bookkeeping.

Absolute value or magnitude. These 10 units of data were also related to the units that were coded as Other. Approximately 6% of the students’ responses (17 of the 279 units of data) from the Open Number Sentences were coded as Other. All of the codes of Other came from Jace’s Individual Sessions. Noticing the commonality between the units of data coded Counterbalance and Other, Counterbalance was refined to include a discussion about this commonality—absolute value or magnitude comparison.

The inclusion of comparisons of absolute magnitude, or absolute value, still fit within the definition because it is still a comparison of two different quantities. Notably, magnitude is distinguished from directed magnitudes, where a quantity is defined by distance and direction. The refined definition uses magnitude that is absolute magnitude. That is, magnitude refers to the use of the absolute value of a quantity, with no distinct direction. For example, when Alice solved 15 + -24 = □, she drew two quantities and used implicit neutralization:

> (Draws 15 tallies with green marker. Draws 24 tallies with pink maker below the 15 green tallies. Then, counts the tallies that do not have green tallies above them. She uses the pink marker to count each of the tallies one by one. Writes -9 in the box.) I did fifteen (points at the green tallies). And then twenty-four negatives (points at the pink tallies). And I had nine negative left.

Alice also used absolute magnitude of comparing 15 and 24. Similarly, the responses from Jace (the units originally coded Other) included similar reasoning to Alice, except his responses used only verbal descriptions and he did not include drawings. For example, when Jace solved 12 + -16 = □, he stated, “Sixteen is greater than twelve. And, sixteen’s the negative number. So sixteen minus twelve equals four. And, it’s got to be negative four because the negative number’s bigger than the regular number.” Here, Jace used the absolute magnitude when he compared 16 and 12 by stating that, “Sixteen is greater than twelve.” He then recognized that this was the absolute magnitude and not the actual number, when he stated that, “the negative number’s bigger than the regular number.” This type of magnitude comparison use was present in all of the units of data coded as Other and was similar to units coded as Counterbalance.

These units of data from Jace that were coded as Other were influential in the refinement of Counterbalance. They were not coded as Counterbalance before because it was not clear that he was employing neutralization of two quantities at first. Because the use of neutralization was implicit from both the students and they were each using two different quantities, it was important to refine this definition. Both Alice and Jace the compared the magnitude, or absolute value, of two different quantities, in all of these units of data and this influenced the refinement of Counterbalance.

Relativity

There were no students’ responses (coded units of data) from the Individual Open Number Sentence Sessions that were coded as Relativity. The only responses that were coded as Relativity came from Jace’s Individual Context Sessions. Of the 28 units of data from Jace’s Individual Context Sessions, there were only 3 units that provided use of integers in relative positions as defined in the original definition of Relativity. Because there were so few units of data, refinement to this model had to be strictly just clarification of wording (see Table 18). The clarification of wording involved clearly defining the role of addition and subtraction and emphasizing the meaning relative numbers.

Table 18

Original and Refined Relativity CMIAS

Original CMIAS Description from Wessman-Enzinger & Mooney (2014)	Refined CMIAS Description
A conceptual model of *Relativity* involves using integers in relative positions and as a comparison to a referent. With relativity, the zero is not actually zero but treated as a referent, or point of reference, for comparison. (p. 203)	*Relativity* is utilized if the integers are used as comparative numbers, otherwise known mathematically as relative numbers. That is, the integers are utilized in a way to describe relative, or arbitrary, positions. With Relativity, the zero represents the point of reference, which may be intentionally or arbitrarily selected. Distinctively, the zero does not represent a quantity of nothing, but is treated as a referent, or point of reference, for comparison. What distinguishes the Relativity from other models is that the actual cardinality of the numbers, or the quantities involved is not necessary. Using the integers as relative numbers in comparison to an unknown referent is the distinguishing feature of Relativity. And, the use of order and integers as relative numbers are a unique feature to Relativity.

Clarification of wording. The first clarification in wording was to define zero as a referent, or point of reference. This was already in the definition, but it was refined to emphasize that the integers are used in comparison to this reference and that this zero as well as the integers do not have cardinality. This was refined by my mathematical perspective and reflection about treating integers as relative numbers. This refinement, although made from my mathematical perspective, was supported by Jace's responses.

For example, Jace exhibited Relativity as he posed a story for the open number sentence, -5 + □ = -7:

> The water levels by the river by Andrew's house was below five, under the ground. And, it got really hot one day and evaporated. So it dropped even more. After all of the water evaporated it was below seven of what it should have been.

When Jace was asked what -5 represented he responded, "That represents what, how high the water levels were before the evaporation." He described that the box represented, "After the, well, when it was evaporating. That was the result." And, he described that -7 represented, "What the reported amount was after the total evaporage." Present in this response was the use of a scale, created by Jace, which is not conventionally used but used the integers to describe the evaporation of a river. Jace did not explicitly define the role of zero; instead, he used the integers in relationship to the height "of what it should have been." In this sense, zero in this situation represents the height of the river, prior to evaporation. The integers in this problem did not have cardinality; rather the integers were in comparison to the level of water prior to evaporation. The integers could be ordered in this context in the order of the different levels of evaporation. This points to the second clarification of wording, which included emphasizing that order is then an important component to this model.

Translation

Approximately 30% of the students' responses to the Individual Open Number Sentences (each coded as a unit of data) exhibited used of the Translation model of thinking. However, the original description of this model did not fully capture the themes present in the students’ responses. Based on my analysis of the data, I made four modifications to the description (see Table 19).

Table 19

Original and Refined Translation CMIAS

Original CMIAS Description from Wessman-Enzinger & Mooney (2014)	Refined CMIAS Description
A conceptual model of *Translation* may be used if integers are treated as vectors, or directed numbers. With this model, integers are used to shift any kind of mathematical object. The zero may represent a zero vector, or no movement. Similar to relativity, the zero can also represent a relative position, with positive and negative numbers representing a movement in one direction or another from the relative zero. (p. 204)	*Translation* is utilized when the integers are treated as vectors or translations. With Translation, the integers are used in ways that shift any kind of mathematical object (e.g., a number, a point, a curve). With Translation, the integers are often treated as vectors moving right or left or up and down a linear model, coordinate plane, or three-dimensional space. The zero in Translation is a zero vector or a translation of no movement. Similar to Relativity, the zero can also represent any arbitrary point with the addition and subtraction of positive and negative numbers representing the Translation in one direction or another from the relative zero. Also, movement and directed distances, or distances with direction, are considered to be Translation. However, sometimes distance may be used without direction. Although it is possible to conceptualize distance without direction, it is still considered to be drawing upon Translation because all distance may be considered as directed. Translation may also be employed with the use of counting strategies because counting fundamentally utilizes movement and order. The distinguishing features of Translation when compared to other models are the idea of order and directed movement.

One change involved clarifying the wording of the description. The other three changes addressed the students' ways of thinking about integers that were not adequately described in the original definition. They pertained to movement as a directed vector, distance without a clear direction, and counting strategies.

Clarification of wording. The first refinement to this model involved the addition of a phrase to provide clarity to the definition. This refinement included adding "right and left or up and down" when describing the use of vectors or directed distance as moving *right and left* or *up and down* a linear model, coordinate plane, or three-dimensional space. For example, 17 of the 85 student responses coded as Translation involved the use of an empty number line or number line drawings. The empty number line or number line drawings came from Kim and Jace only (6 units from Kim and 11 units from Jace); none of the units of data from Alice contained empty number lines or number lines. Accompanied within these units of data of empty number lines or number line drawings, there were verbal descriptions with movement, which were coded as Translation. However, the students did not draw their empty number lines or number lines consistently. That is, sometimes the empty number lines or number lines were horizontal and sometimes they were vertical. Sometimes the negative numbers were on the left of the horizontal number line; sometimes the negative numbers were on the right of the horizontal number line. Sometimes the negatives were on the upper portion of the number line; sometimes the negatives were on the lower portion of the number line. Because the students applied translations and distances without consistency in the placement of negatives, this affected the translations they made along their number lines (adding sometimes being left, right, up, or down the number line). This definition was refined to incorporate the students' flexible use of movement left, right, up, or down.

Movement as a directed vector. There were 67 of the 85 units that were initially coded Translation that supported the use of directed vectors or a shift on a linear space, which was present in the initial definition of Translation. However, these students' responses also incorporated "movement" which was not previously clear in the definition. Shifts, which are directed vectors, are in the initial definition; however, the definition needed to be modified to include the use of "movement." For example, Jace before applying a shift from -7 to -2 when solving -7 + □ = -2 stated that, "You have to, and you want to go lower" to solve this. Kim referenced this movement as she used Translation, "And, it was so strong it blew past negative nine (waves hand to the right), it blew past zero, and it stopped at 8." Similarly, Alice used Translation and also referenced movement when she justified her translation or directed vector, with, "because you are going up" when she solved -4 + □ = 10.

Distance without a clear direction. Directed distance fit the initial definition of Translation with the description of shifts; but distance without a clear direction was not adequately addressed in the initial definition. As part of this refinement, discussion was provided about distance that is conceptualized without direction but is still directed. For example, when Kim solved 12 – 18 = □, she stated, "So once you get to zero you have six left over. So you just keep going (waves hand to the left) to negative six." Although this used movement, which was added to the definition, this use of integers aligned well with the initial CMIAS description of Translation because she used a shift in the same direction (i.e., a directed vector). However, when Jace solved -6 + □ = 15, he drew a number line with two distances. He first drew a distance from -6 to 0 and then a distance from 15 to 0. In this sense, Jace did not have a clear, singular "directed" vector. Instead, he had a directed distance going left to right and a second directed distanced going right to left towards zero. He stated, "So, that would be negative six right here and fifteen right here. It would be fifteen [from fifteen to zero] plus another six [from zero to negative six]…Just regular twenty-one." The challenge with the initial CMIAS description was that Jace, although drawing upon distance which can be thought of as movement along a number line, did not have a clear directed vector from -6 to 15. However, this type of response seemed intimately related to recognizing directed vectors, since Jace knew to add the distances of 6 and 15 to determine the solution of 21 and he created two distances that were directed towards zero. Although Jace's use of movement here is similar to Kim's because each included movement, he used movement and distance differently. He did not explicitly talk about movement and he used multiple distances directed to zero in different directions, indicating a possible use of undirected distance or multiple directed distances. There were 7 units of data like this that included distance without a clear direction, out of the 85 units coded as Translation in Phase 2. For this reason, it was determined Translation needed to be refined to support this type of reasoning. Translation was refined to include a description about distance that is not recognizably directed from the student.

Counting strategies. Counting was used to enact a translation and was also paired with movement. There were 18 units of data in the 85 units that were coded as Translation that included counting strategies. For example, when Kim solved -6 + □ = 15, she counted:

> I started off with negative six. Then I was like negative five (thumb), negative four (next finger), negative three (another finger), negative three (another finger), negative one (another finger), zero (thumb on other hand). Then I counted five: one, two, three, four, five (holds up only four more fingers). Wait, one, two, three, four, yeah. Five (holds up thumb on left hand again). That was eleven. And then, since I was at five and I needed to get to fifteen, I added ten more onto eleven and I got twenty-one.

Kim used Translation as she made a shift from -6 to 15 to solve this. However, she used counting which was not present in the previous definition of Translation. Similarly, Alice used counting as she also solved -6 + □ = 15. She counted:

> Well, I did six lines at first representing negative six. Then I did six, five, four, three, two, one (points at each tally mark), zero, one two, three four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen (continues pointing at each tally). And, then I had sixteen lines.

Because the students used counting to support using a translation, this was added to the refined definition.

The Expansion of Rule to Proceduralization, Analogy, and Algebraic Reasoning

Rule was the most frequently utilized code. For example, 59 units of data out of 93 (63.4%) from Alice's Individual Open Number Sentence Sessions were initially coded as Rule, 76 units of data out of 93 (81.7%) from Kim's Individual Open Number Sentence Sessions, and 83 units of data out of 93 (89.2%) from Jace's Individual Open Number Sentence Sessions were initially coded as Rule. Thus, of these 279 units of data from Alice, Kim, and Jace's Individual Open Number Sentences, 218 units of data or 78.1% of the units were initially coded was Rule.

Because Rule was a code that was being used at such a high overall percentage (78.1%), the original definition for this conceptual model was not adequately capturing the nuances in the students' thinking. Rule needed either refinement or expansion. My analysis of each of these units of data resulted in the identification of three main themes that characterized the students' thinking: analogies, algebraic reasoning, and procedures. These themes correspond to the changes I made to the definition of the conceptual model (see Table 20). Rule was renamed as Proceduralization and the definition was refined and two new conceptual models were developed, Analogy and Algebraic Reasoning

Table 20

Original and Refined Rule CMIAS

Original CMIAS Description from Wessman-Enzinger & Mooney (2014)	Refined CMIAS Description
Rule	**Proceduralization**
A conceptual model of *Rule* occurs when integers are used in a way that is contingent to some outside "rule" or algorithm. This model may be constructed outside the context of the problem or task. For example, several students applied an algorithm for subtracting a negative number called "keep-change-change," writing a story that involved addition instead of subtraction. (p. 204)	*Proceduralization* is utilized when integers are used in a way that is contingent to a rule, procedure, or algorithm. Zero does not have specifically defined meaning in this model and the integers do not necessarily have specific roles in terms of addition and subtraction. Rather, Proceduralization represents thinking that draws upon self-invented or conventionally utilized rules about operations with integers that are made explicit or used implicitly.
	Analogy
	Analogy is utilized when the integers are used in a way that connects an integer

Table Continues

Original CMIAS Description from Wessman-Enzinger & Mooney (2014)	Refined CMIAS Description
	addition and subtraction number sentence or context to a different integer addition and subtraction number sentences or context. Within Analogy there are two subcategories: Whole Number Analogy and Integer Analogy. Whole Number Analogies are comparisons of integer addition or subtraction problems to other addition or subtraction problems with only whole numbers. For example, -2 + -3 = □ may be solved by comparison to 2 + 3 = 5. Integers Analogies are comparisons of integer addition or subtraction problems to different, and not necessarily equivalent, integer addition or subtraction problems. For example, 3 – -2 = □ may be solved by comparison to -2 – 3 = -5. Similar to Proceduralization, zero doesn't have a specifically defined meaning in this model. However, what distinguishes this model from other is the comparison of addition and subtraction problems to other addition and subtraction problems. **Algebraic Reasoning** *Algebraic Reasoning* is utilized when the integers are used alongside algebraic properties, such as inverse or commutative properties of addition and subtraction. Algebraic reasoning is also used with integer addition or subtraction when the integer addition or subtraction number sentence, equation, or expression is re-expressed as a number sentence, equation, or expression or equation that is mathematically equivalent. The meaning of zero in Algebraic reasoning is that of the additive identity (e.g., a + 0 = a) which is produced from integer inverses (e.g., a + -a = 0).

Proceduralization

Of the 218 units of data that were coded Rule, 132 of these units of data included the use of procedures that was different from the use of an analogy or algebraic reasoning. That is, approximately 61% of the data supported the use of a procedure, different from making an analogy or changing the structure of a problem. Because new models were generated from Rule, Rule had minimal refinements. Based on my analysis of Rule, besides the emergence of two new models, only two minor refinements were made. The first refinement included a name change and the second refinement included clarification of wording.

Name change. The first refinement to this model was to change the name from Rule to Proceduralization. Although the use of the word rule is still present in the definition. This was changed because Rule may evoke other cultural connotations than what was intended this model, like an algebraic rule. This change was not made out of the data, but rather at my discretion. As a field we talk procedural knowledge and ideas like procedural fluency; thus, the name was of the model was changed from Rule to Proceduralization to embrace the definition of the conceptual model better. Despite this name change, Proceduralization is related to the previously defined Rule.

Clarification of wording. The other change to this conceptual model was a wording issue. Student's word use of "keep-keep-change" was omitted from this definition. This decision was made because in none of the units of the data did the students in this study use phrase, "keep-keep-change." Since "keep-keep-change" seemed to be something not developed by students (no units of data supporting this) and is not something I hope to advocate for, even if unintentionally by inclusion in the definition, it was omitted.

Although the students in this study didn't use that language, the students did develop their own procedures and use procedures often, so leaving the definition as it was was good idea. When Jace solved -5 – 4 = □, he used a procedure or rule about the minus sign:

> Because it's subtraction, you don't have to keep this (crosses of the minus sign of -5). But, never, to make it easier, you would have it keep it right away. So fine minus four would equal one, but that would be negative one. You drag this over here (points at -5, then draws a line connecting from -5 to the box). Five minus four is one.

In this example provided by Jace, zero and integers do not have clearly defined meanings. This was the last refinement to the definition. Of the 279 units of data from the Individual Open Number Sentences, 146 units of data from all three of the students included the use of a procedure like described above. Although the procedures were not always like the one highlighted above, all of the rules and procedures were not dependent upon the role of zero or make use of the integers in a clearly defined way.

Analogy

Of the 218 units of data that were coded Rule, 119 of these units of data included the use of an analogy, or comparing a number sentence to a different number sentence. That is, approximately 55% of the data supported the use of analogies, which was justification for a new conceptual model (units sometimes supported more than one CMIAS). This new definition was developed based on the units of data that I analyzed.

Definition. *Analogy* is utilized when the integers are used in a way that connects an integer addition and subtraction number sentence or context to a different integer addition and subtraction number sentences or context. Within Analogy there are two subcategories: Whole Number Analogy and Integer Analogy. Whole Number Analogies are comparisons of integer addition or subtraction problems to other addition or subtraction problems with only whole numbers. For example, -2 + -3 = □ may be solved by comparison to 2 + 3 = 5. Integers Analogies are comparisons of integer addition or subtraction problems to different, and not necessarily equivalent, integer addition or subtraction problems. For example, 3 – -2 = □ may be solved by comparison to -2 – 3 = -5. Similar to Proceduralization, zero doesn't have a specifically defined meaning in this model. However, what distinguishes this model from other is the comparison of addition and subtraction problems to other addition and subtraction problems. Based on my analysis of the data, the two main features of these definition is the comparison of different number sentences and

Comparing different number sentences. All of the units of data that were considered Analogy exhibited a comparison of two different number sentences. For example, when Alice solved -5 – □ = 0 she wrote -5 in the box and stated, "Because if you have five and you got rid of five, that would mean that you had zero left." Here Alice compared -5 – -5 = 0 to 5 – 5 = 0. This was initially coded as Rule because it seemed as if Alice was drawing upon an implicit rule that she could use the computation 5 – 5 = 0. The abundance of responses like this inspired the development of this new CMIAS.

Whole number and integer analogies. The definition also includes a description of the two sub-groups of Analogy: Whole Number Analogy and Integer Analogy. This component of the definition was inspired by Jace's units of data where he compared the open number sentence -10 – □ = -11 to -10 – -1 = -9. He stated, "Hmm. I'm not really sure about this one either. Because if you do negative ten minus negative one, then that would get you negative nine." In this same unit of data, he compared -10 – □ = -11 to -10 – 4 = -6. Although -10 – 4 = -6 is not correct, Jace made an analogy to a different number sentence, which is how all of the units in this CMIAS are defined. Similarly, when Jace solved 3 – □ = 4, he made this different type of analogy again. He compared 3 – □ = 4 to 3 – 7 = -4 to make sense of the open number sentence. He stated, "I did three minus seven, but that would equal negative four instead of just four." Jace continued as he struggled to solve 3 – □ = 4 by also making an analogy to 3 – -1, "So here's three (points at 3). Minus negative one, would just get you two." Analogies to number sentence like 3 – -1 = □ were different because they involved negative integers, whereas most analogies in the units of data (and also the analogies we see in the literature on integers) clearly involved only whole number numbers.

There were not many units of data that had Integer Analogies. For example, of the 93 units of data from Jace's Individual Open Number Sentence Sessions, only 7 of these units of data contained analogies where a number sentence with negative integers was compared to another non-equivalent number sentence with negative integers (i.e., Integer Analogy). Kim did not appear to have units of data to support the Integer Analogies. And, Alice had 2 units of data from the 93 units of data from her Individual Open Number Sentences Sessions that supported Integer Analogies. For example, when Alice solved -11 – -2 =□ she first compared it to 11 – 2 = 9 (i.e., Whole Number Analogy), but then Alice compared -11 – -2 = □ to -11 – -12 = 1 (i.e., Integer Analogy). Comparing -11 – -2 to 11 – 2 = 9 is different than comparing -11 – -2 = □ to -11 – -12 =1. Yet, both of these are comparisons, or analogies, to different number sentences (i.e., -11 – -2 ≠ 11 – 2 ≠ -11 – -12). Because there were so few units of data, it didn't warrant a new conceptual model, but this was captured in the definition.

Algebraic Reasoning

Of the 218 units of data that were coded Rule, 62 of these units of data included the re-expressing of a number sentence through the use of the commutative property or inverses. That is, approximately 28% of the data supported the use of algebraic reasoning, which was justification for a new conceptual model. This new definition was developed based on the units of data that I analyzed. All of the units of data supported re-expressing an equation or open number sentence based on an algebraic property.

Definition. *Algebraic Reasoning* is utilized when the integers are used alongside algebraic properties, such as inverse or commutative properties of addition and subtraction. Algebraic reasoning is also used with integer addition or subtraction when the integer addition or subtraction number sentence, equation, or expression is re-expressed as a number sentence, equation, or expression or equation that is mathematically equivalent. The meaning of zero in Algebraic reasoning is that of the additive identity (e.g., a + 0 = a) which is produced from integer inverses (e.g., a + -a = 0).

Re-expressing an equation or open number sentence. Using an algebraic property to re-express an equation or open number sentence is based upon using an algebraic property, like commutative property or inverses. For example, when Alice solved -10 – □ = -11 she stated, "Because if we were going backwards, negative eleven (points at -11) plus (points at equal sign) one (points at 1) would equal (points at minus sign) negative ten (points at -10).” In this example, Alice explicitly referenced solving the problem “backwards” that is solving -11 + 1 = -10. Here, Alice re-expressed -10 – □ = -11 as -11 + □ = -10, which drew upon addition as the inverse operation of subtraction. Similarly, students also referenced use of the commutative property. For example, Jace solved -3 + □ = 14:

> Because negative three is basically box (points at box) minus three. So, I did fourteen plus three and I got seventeen. And, seventeen minus three (points at -3) equals fourteen (points at fourteen). It’s kind of like the commutative property.

In this excerpt, Jace solved -3 + □ = 14 by re-expressing it as 14 + 3 = □. In this excerpt, he even explicitly referenced using the “commutative property” to solve the open number sentence. These examples highlighted by Alice and Jace captured the essence of re-expressing an equation or open number sentence that was present in all of the units of data that supported developing a new conceptual model. Alice’s re-expression of the open number sentence drew upon utilizing addition and subtraction as inverse operations; and, Jace’s re-expression of his open number sentence involved use of the commutative property of addition.

Summary

The five original CMIAS (i.e., Bookkeeping, Counterbalance, Relativity, Translation, Rule) were modified to seven CMIAS (i.e., Bookkeeping, Counterbalance, Relativity, Translation, Proceduralization, Analogy, Algebraic Reasoning). These new definitions are summarized in Table 21. The descriptions for each of these conceptual models were either refined or have a new definition generated.

Table 21

The Refined Definitions of CMIAS

The Refined Definitions of the CMIAS
Bookkeeping *Bookkeeping* is utilized when the integers are used in a way to describe losses and gains of quantities. The zero in the bookkeeping conceptual model represents having neither a specific gain nor loss. In Bookkeeping, the positive and negative integers represent a gain or loss of something applied to a singular quantity. That is, the addition of a positive integer may be treated as gain to the singular quantity and the addition of a negative integer may be treated as a loss to the singular quantity. Or, the subtraction of a positive integer may be treated as a loss to the singular quantity and the subtraction of a negative integer may be treated as a gain to the singular quantity. Bookkeeping points to thinking that is mostly quantitative reasoning without explicit reference to order and without reference neutralization. Although the positive integers are treated as an increase in the quantity and the negative integers are treated as loss in quantities, the addition of positive integers to a negative quantity may also be considered a loss of negative quantity. A distinguishing element of Bookkeeping compared to other models is that the gain or loss is being applied to a singular quantity.
Counterbalance *Counterbalance* is utilized when the negative and positive integers are treated as two separate quantities in a way that balances each other out. The zero in the counterbalance conceptual model represents a status of neutralization. Positive and negative numbers in Counterbalance are not just opposites, but opposites that neutralize. Addition of integers, whether positive or negative, represents joining quantities where the equal number of positive and negative integers are neutralized and the sum is the integers not neutralized. Subtraction of integers, whether positive or negative, represents removing quantities, which may entail removing quantities from a status of neutralization. Similar to Bookkeeping, Counterbalance points to thinking that is mostly quantitative. A distinguishing feature of Counterbalance from other models, and specifically Bookkeeping, is that there are two quantities that always remain present in the Counterbalance, even when neutralized. The absolute value, or magnitude, of these two different quantities may be compared to solve addition and subtraction problems.
Relativity *Relativity* is utilized if the integers are used as comparative numbers, otherwise known mathematically as relative numbers. That is, the integers are utilized in a way to describe relative, or arbitrary, positions. With Relativity, the zero represents the point

Table Continues

of reference, which may be intentionally or arbitrarily selected.

Distinctively, the zero does not represent a quantity of nothing, but is treated as a referent, or point of reference, for comparison. What distinguishes the Relativity from other models is that the actual cardinality of the numbers, or the quantities involved is not necessary. Using the integers as relative numbers in comparison to an unknown referent is the distinguishing feature of Relativity. And, the use of order and integers as relative numbers are a unique feature to Relativity.

Translation
Translation is utilized when the integers are treated as vectors or translations. With Translation, the integers are used in ways that shift any kind of mathematical object (e.g., a number, a point, a curve). With Translation, the integers are often treated as vectors moving right or left or up and down a linear model, coordinate plane, or three-dimensional space. The zero in Translation is a zero vector or a translation of no movement. Similar to Relativity, the zero can also represent any arbitrary point with the addition and subtraction of positive and negative numbers representing the Translation in one direction or another from the relative zero.

Also, movement and directed distances, or distances with direction, are considered to be Translation. However, sometimes distance may be used without direction. Although it is possible to conceptualize distance without direction, it is still considered to be drawing upon Translation because all distance may be considered as directed. Translation may also be employed with the use of counting strategies because counting fundamentally utilizes movement and order. The distinguishing features of Translation when compared to other models are the idea of order and directed movement.

Proceduralization
Proceduralization is utilized when integers are used in a way that is contingent to a rule, procedure, or algorithm. Zero does not have specifically defined meaning in this model and the integers do not necessarily have specific roles in terms of addition and subtraction. Rather, Proceduralization represents thinking that draws upon self-invented or conventionally utilized rules about operations with integers that are made explicit or used implicitly.

Analogy
Analogy is utilized when the integers are used in a way that connects an integer addition and subtraction number sentence or context to a different integer addition and subtraction number sentences or context. Within Analogy there are two subcategories: Whole Number Analogy and Integer Analogy. Whole Number Analogies are comparisons of integer addition or subtraction problems to other addition or subtraction problems with only whole numbers. For example, $-2 + -3 = \square$ may be solved by comparison to $2 + 3 = 5$. Integers Analogies are comparisons of integer addition or subtraction problems to different, and not necessarily equivalent, integer addition or

Table Continues

subtraction problems. For example, 3 – -2 = □ may be solved by comparison to -2 – 3 = -5. Similar to Proceduralization, zero doesn't have a specifically defined meaning in this model.
However, what distinguishes this model from other is the comparison of addition and subtraction problems to other addition and subtraction problems.

Algebraic Reasoning
Algebraic Reasoning is utilized when the integers are used alongside algebraic properties, such as inverse or commutative properties of addition and subtraction. Algebraic reasoning is also used with integer addition or subtraction when the integer addition or subtraction number sentence, equation, or expression is re-expressed as a number sentence, equation, or expression or equation that is mathematically equivalent. The meaning of zero in Algebraic reasoning is that of the additive identity (e.g., a + 0 = a) which is produced from integer inverses (e.g., a + -a = 0).

Students' Use of CMIAS

The purpose of this section is to illustrate the use of the CMIAS by Alice, Jace, and Kim. The discussion above gave mathematical insight into these uses, and this section will give insight into the ways these students used the CMIAS.

Bookkeeping

Although it may be expected that the context of money with borrowing or gaining of money may best work with Bookkeeping, the focus of this model is that the positive and negative integers represent a gain or loss of something applied to a singular quantity and do not necessarily require the use of money as the context. For example, gains and losses can be conceptualized with "owing and gaining of candy bars" or "wanting and receiving of baseball cards." An example of the Bookkeeping illustrated by Alice for the number sentence -12 + 7 = □, "I need twelve markers, or I need twelve pencils. I got seven. Now, I need five." Here the "loss" is presented by the negative integer is represented by the "need" of twelve markers. The positive integer and "gain" is represented by the acquisition of seven markers.

When students used Bookkeeping there was not explicit focus on order or use neutralization, but use of integers as gains and losses. Alice demonstrated Bookkeeping when she provided the following solution for -20 + 15 = □:

> (Starts drawing boxes) I have fifteen boxes. (Continues drawing 15 boxes, counts the fifteen boxes). Fifteen and ...Well, actually I have twenty boxes (adds five more boxes to drawing so that she has twenty boxes). Then, I just take fifteen more. (Crosses off boxes, stops, counts five from the right to the left, and crosses off two more boxes to cross off a total of 15 boxes). So that add, that would give me negative five because I had negative twenty and I got fifteen more, and so now I have negative five.

Figure 11. Bookkeeping Open Number Sentence Example.

Here the addition of a positive integer was treated as a loss of the negative integers. That is, Alice represented -20 as a quantity of 20 boxes. Then, to address adding 15, she crossed off 15 boxes, or applied a loss of -15. Although Alice did not symbolize it as such, a mathematical interpretation of Alice's strategy is -20 – -15 = -20 + 15. The use of Bookkeeping, which is defined as the pairing into two different groups, "gains" and "losses," where the gain or loss is applied to a singular quantity, was not used in traditional ways of thinking of addition as gains. For example, addition was often treated as a loss of negative items, as shown in Figure 11. Although Bookkeeping could include the addition as the gain of discrete items or subtraction as the losses as taking discrete items, mostly students used addition as a loss of negative quantities applied to a singular quantity. This differs from Counterbalance, which will be discussed next, where there are often two different quantities represented or used.

Overall, Alice used Bookkeeping 20% of the time when she solved open number sentences in the Individual Open Number Sentence Sessions (19 of the 93 open number sentences). Alice also used Bookkeeping 100% of the time when she posed stories for open number sentences in the Individual Context Sessions (23 of the 23 stories posed). Both Kim and Jace did not use Bookkeeping when they solved open number sentences in the Individual Open Number Sentences. Kim used Bookkeeping to pose stories for open number sentences 75% of the time in the Individual Context Sessions (18 of the 24 stories posed). And, Jace used Bookkeeping to pose stories for open number sentences 71% of the time in the Individual Context Sessions (20 of the 28 stories posed).

Counterbalance

Although none of the students used Counterbalance to pose stories, students did draw upon Counterbalance to solve open number sentences. When students utilized solutions or strategies that drew upon Counterbalance, they did not explicitly say "neutralization," however there are implicit "neutralizations." For example, Alice often used her hand to cover up both positive and negative tallies in a drawing, recognizing that the leftover that was not covered by her hand was the result of the addition and subtraction. The act of using her hand to cover up the tallies was an act of "neutralizing" the negatives without explicitly verbalizing it. Counterbalance use is demonstrated in the solution below as Alice solved 15 + -24 = □. Alice first drew 15 tallies with a green marker (see Figure 12). She then drew 24 tallies with a pink marker directly below the 15 green tallies. Then, she counted the pink tallies that did not have green tallies above them. She counted the tallies one by one and wrote -9 in the box. When probed to explain what she did verbally, Alice responded, "I did fifteen (points at the green tallies). And then twenty-four negatives (points at the pink tallies). And I had nine negative left." After she explained that the green tallies represented 15 and the pink tallies represented -24 Alice shared, "Because there was more, there was more not negatives. There was negatives left (uses right hand to point at the pink tallies that do not have green tallies above them)." Alice utilized neutralization as she ignored the green and pink tallies that were "neutralized" and she counted the "leftovers."

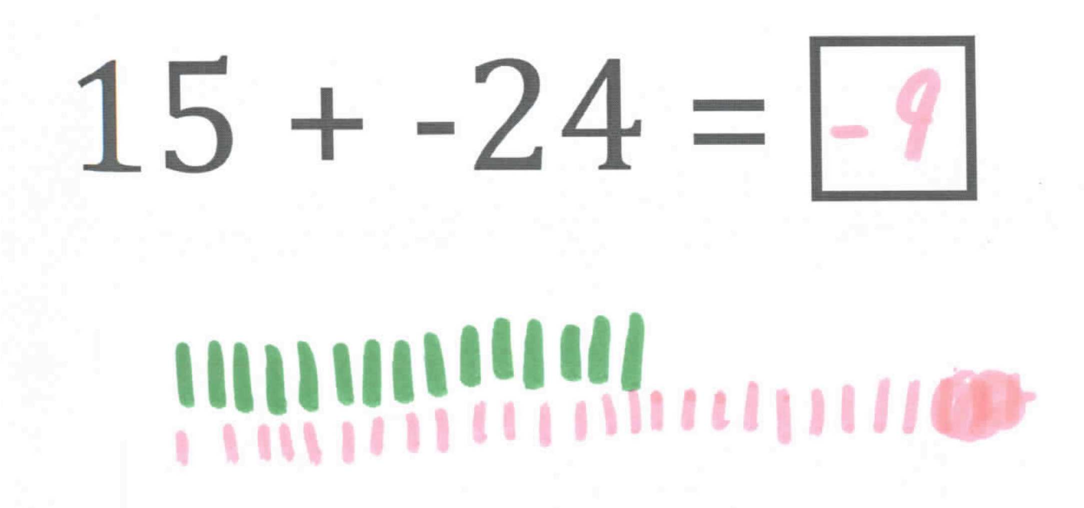


Figure 12. Counterbalance Open Number Sentence Example.

Overall, Alice used Counterbalance 38% of the time when she solved open number sentences in the Individual Open Number Sentence Sessions (35 of the 93 open number sentences). Kim used Counterbalance 10% of the time (9 of the 93 open number sentences) when she solved open number sentences in the Individual Open Number Sentence Sessions. Jace used Counterbalance 14% of the time (13 of the 93 open number sentences) when she solved open number sentences in the Individual Open Number Sentence Sessions. Neither Alice, Kim, and Jace used Counterbalance to pose stories in the Individual Context Sessions.

Relativity

Although the students did not appear to use Relativity when they were solving open number sentences, one of the students, Jace, used Relativity to pose stories for open number sentences. Jace used Relativity to pose the following story for the number sentence $-5 + \square = -7$:

> There was a number line. And the number line said ten through negative ten. And they tried to figure out how bright the sun was. And realized it was five measures shorter than what it should have been. The next day they wanted to, they figured out that it dropped two measures. Or, that it dropped so many measures. And after they recorded that they figured out that it was seven measures below what they thought it would have been.

In this story, the integers are being treated as relative numbers that describe the measure of brightness of the sun. Although we know that -5 represents “five measures shorter than what it should have been,” we do not know the actual brightness of the sun. Using the integers as relative numbers in comparison to an unknown referent is a distinguishing feature of Relativity.

Alice, Kim, and Jace did not use Relativity when they solved open number sentences in the Individual Open Number Sentence Sessions. Neither Alice nor Kim used Relativity to pose stories in the Individual Context Sessions. However, Jace used Relativity 11% of the time to pose stories in the Individual Context Sessions (3 of the 28 stories posed).

Translation

The following story, posed by Kim, represents the number sentence, -2 – 3 = □ and provides an example of her use of Translation:

> It was negative two degrees out and then it soon dropped three more degrees. Which made it negative five.

Here the -2 is a relative number representing a temperature, that is translated to another relative number, or temperature.

Kim used Translation, with drawing a number line, to determine that the solution to the open number sentence, □ – -2 = 1. First, Kim thought the answer was 3 and then changed her mind to -1. When asked "How are you thinking of it as negative one?" Kim responded:

> The answer was one (points at 1 in the number sentence) and here was a negative two (points at -2). So I sort of knew the only way I could get to a positive, which was the one (points at 1), which was to like have a smaller negative number (points at -2) besides 0 and then negative two. And, the only number was negative one (points at box with -1). And, if you did it, it was like a couple back in when I

had the negative -9. It's pretty much just like that. So, the negative one, the two (starts drawing a number line). This will be the biggest (marks the number line with 1 and then 0). I don't know why I didn’t make this smaller they are close numbers. (laughs) I will go with. I will go with negative two I think (draws -2 on the number line) and that should work. No (crosses off -2 on number line). One (marks -1 on the number line). When you subtract the two off of it, it would go, but when it hit zero it's lost one (marks number line). So, it has zero. It has one remaining over, so you could just add onto and go into the positive area. And it, when you got done using your remainders it'd be one.

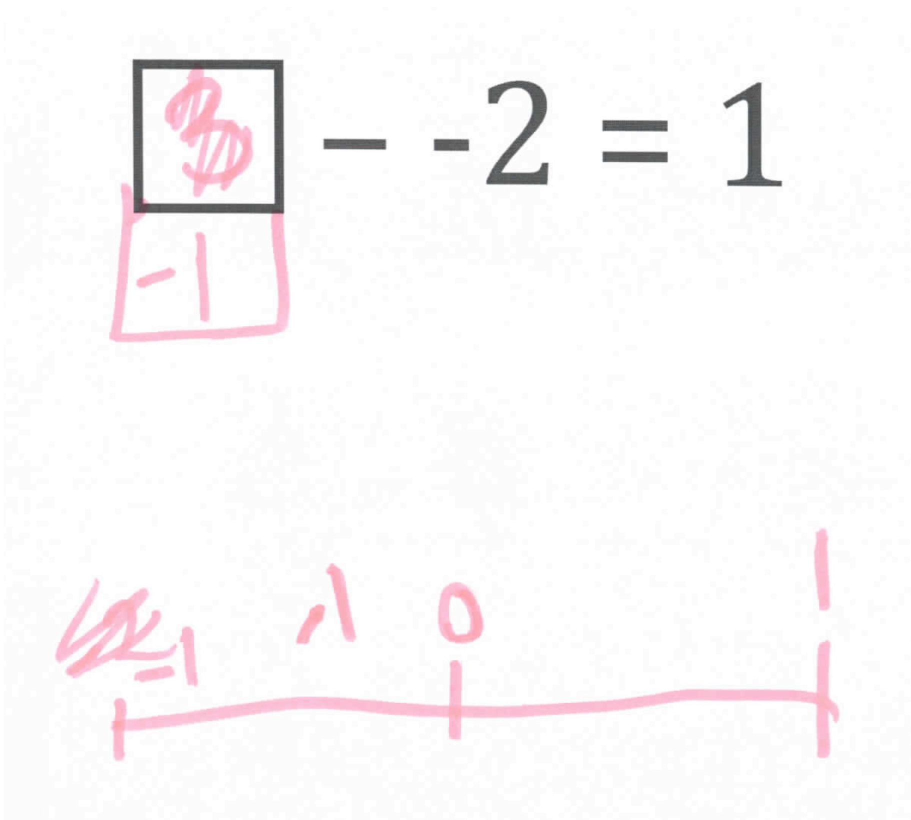


Figure 13. Translation Open Number Sentence Example.

Here Kim starts at -1 interprets the subtracting of -2 as a translation to the right to 1. She references that going from -1 to 1, which is a use of Translation, and then from 0 to 1, which again is a second use of Translation. Although Kim used Translation with a number line drawing (see Figure 13), it is possible to use Translation without a number line. For example, when Kim solved -5 – 4 = □, she used Translation, without a drawing:

> I think that's when you do deeper into the negatives. Like higher, as in, let's say you start out with just positive fifteen and it going to go higher past a hundred and stuff that's how it goes higher. So, with the negatives, negative fifteen, it's going to go like negative one hundred. It's just going in complete different directions (waves hand to the left)… The positives go way (waves hand to the right), way that way forever and ever. And, the negatives go that way (waves hand to the left), way forever that way… so that just made me feel like it's going to be like, oh it's going to be a negative, it's going to go (waves hand to the left) a little bit farther that way.

Counting strategies were considered to be utilizing Translation. For example, Alice counted from 5 to -3 to solve 5 + □ = -3.

A: I did five minus blank to give you three. And then I counted till I got, I counted from five to until I got three and I got negative eight.

T: Can you help me understand what you mean you counted? Can you maybe do it aloud so I can understand?

A: Five, four, then I did four, three, two, one, zero, one, two, three. And then I got eight.

Even though Alice did not explicitly count with negative integers and only stated positive integers, she continued counting after zero in a way symmetric and similar to the negative integers. She used her fingers with each count after five to determine that, " I got 8" and she wrote -8 in the box.

Overall, Alice used Translation 39% of the time (36 of the 93 open number sentences) when she solved open number sentences in the Individual Sessions. Alice did not use Translation to pose stories in the Individual Context Sessions. Kim used Translation 54% of the time (50 of the 93 open number sentences) when she solved open number sentences in the Individual Open Number Sentence Sessions. Kim used Translation to pose stories for open number sentences 25% of the time (6 of the 24 stories posed) in the Individual Context Sessions. Jace used Translation 25% of the time (23 of the 93 open number sentences) when she solved open number sentences in the Individual Open Number Sentence Sessions.

Proceduralization

None of the students used Proceduralization to pose stories in this study. However, the students used Proceduralization to solve open number sentences. Many times in this study students would incorrectly ignore the second minus symbol, either explicitly stating so or not verbalizing it. Jace used Proceduralization, when he created a rule about the minus symbols that was mathematically incorrect, when he solved the problem □ – -4 = 0 (see Figure 14):

> (Draws a vertical problem.) So it says the answer is zero, and this number is four (points at -4). That means that, well this number is negative four (points at negative four). So that means that the first number has to be four too because four

minus four equals zero. And, we don't need that (uses marker to cross of minus sign) and we just need this (circles the negative symbol).

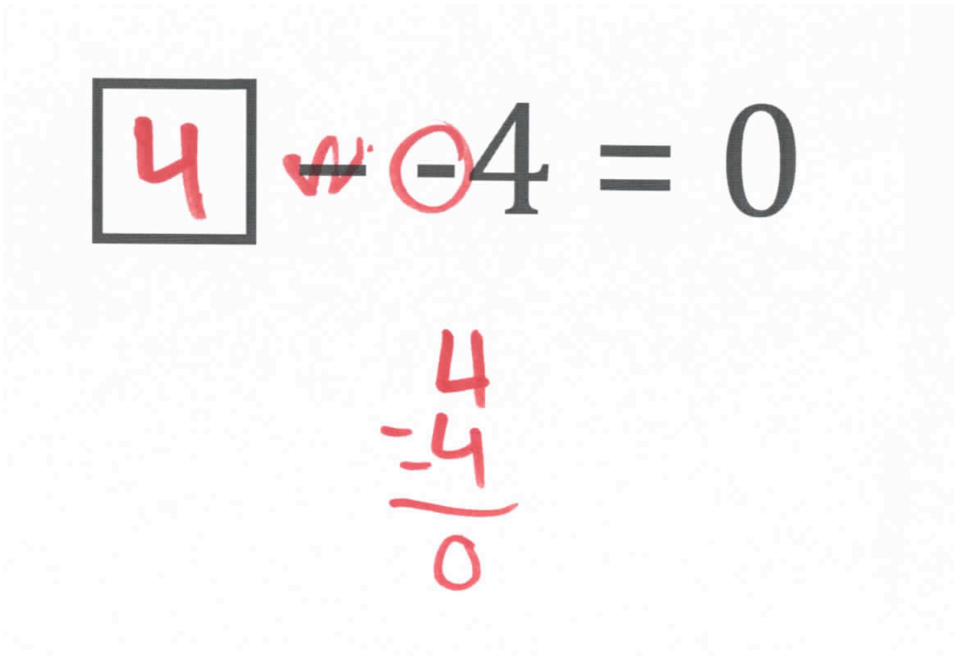


Figure 14. Proceduralization Open Number Sentence Example.

However, some Proceduralization proved to be productive. Alice demonstrated a mathematically correct Proceduralization when she solved 2 – -3 = □ by placing 5 in the box. When asked how she knew what the correct answer was:

A: You do plus (changes minus sign to plus sign) and take that off (scratches off the negative symbol of -3). And, it just be like two plus three.

T: Ok. So, how could you explain that to somebody that maybe ... like the rest of your class hasn't had the experiences that you guys had. So, how might you convince somebody that that's true?

A: Well, they would think it's two because they don't know the right answer.

T: (laughs) But, what if they were like I don't believe you. How might you try to convince them?

A: I'll say that... I'll show how I figured it out. Like a two (draws two tallies). I don't know (laughs).

Here Alice used a Proceduralization to solve the problem, got the correct answer, but was unable to solve it another way.

Students utilized Proceduralization when the students used a generalization to solve the problem or knew the solution to a problem immediately by using their rule. Jace solved 12 + -16 = □ using Proceduralization with a rule he created, "Well, if there was ... Because it's a negative number it's pretty much saying ...well, you don't need the addition symbol. So it would be sixteen minus twelve, which (draws 16 - 12 = 4) sixteen minus twelve would be four."

Overall, Alice used Proceduralization 47% of the time when she solved open number sentences in the Individual Sessions (44 of the 93 open number sentences). Kim used Proceduralization 49% of the time when she solved open number sentences in the Individual Open Number Sentence Sessions (46 of the 93 open number sentences). And, Jace used Proceduralization 60% of the time when she solved open number sentences in the Individual Open Number Sentence Sessions (56 of the 93 open number sentences). Neither Alice, Kim, nor Jace used Proceduralization to pose stories in the Individual Context Sessions.

Analogy

Students did not appear to use Analogy when generating stories. However, students used Analogy as they solved integer addition and subtraction number sentences.

Kim utilized Analogy, and specifically a Whole Number Analogy, when she solved -12 + -5 = □ in the following excerpt:

K: (Writes -17 in the box.) Both of the numbers have the negative sign in front of them. That means that they are both negatives. And, that's pretty much the same thing as twelve plus five when you add a negative twelve plus five. It's just negatives this time. So you just (shrugs) ... the twelve plus five and got seventeen. And I just added the negative onto it.

T: How come you think negative twelve plus negative five is similar to twelve plus five?

K: It comes out as the exact same answer, but the only difference is that they add a negative sign to it.

In this transcript excerpt, Kim stated that -12 + -5 = □ is "pretty much the same" as 12 + 5 = □. Kim's direct comparison of -12 + -5 = □ to 12 + 5 = □ is how to identify the use of Analogy.

Students also used Analogy, and specifically Integer Analogy, when an integer addition or subtraction problem was compared to another addition or subtraction problem. For example, throughout the Individual Sessions Jace often reasoned that the answer to problem types like -5 – 4 was -1. In one instance, he changed his answer by using an Integer Analogy. He made an Integer Analogy when he compared -5 – 4 = □ to 4 – -5 = □. In the following excerpt, as Jace struggled to solve -5 – 4 = □, he compared -5 – 4 = □ to 4 – -5 = □ to help make sense of the problem:

J: Like I'm thinking maybe ... I'll write it down, but I think that negative five minus four (start drawing horizontal problem) would equal negative one. Because, just like last problem, it's a negative number and a subtraction. So, but, I don't know.

T: What's got you kind of questioning yourself right now?

J: Because... ah maybe not. You know what... I think it's nine because ...if you have a negative five and you flip the problem around. So four minus negative five that would be five because you are taking away a negative number from the four even though you don't have a negative number. So, it would be plus instead of minus a negative. I'll wait a second.

T: Alright so you first thought it was negative one.

J: mmm-hmm.

T: And, now you don't think it is anymore? What made you think that it's not that anymore?

J: Because it's ... to be negative one it would have to be negative five minus negative four, because five minus four equals one and then they're all negative numbers. But, since it's not then it's going to be a different answer.

T: Ok. So then you wrote four minus negative five equals nine (points at second horizontal problem).

J: Mmm-hmm.

T: How come you switched the order (points at 4 – -5)?

J: Because I think it helped me understand it better.

This excerpt highlighted Jace's struggle with -5 – 4 = □, and how he compared -5 – 4 = □ to 4 – -5 = □ because it "helped me understand better." Although he did not get the correct answer, he started to think about -5 – 4 = □ differently than he had before.

Overall, Alice used Analogy 24% of the time when she solved open number sentences in the Individual Sessions (22 of the 93 open number sentences). Kim used Analogy 61% of the time when she solved open number sentences in the Individual Open Number Sentence Sessions (57 of the 93 open number sentences). And, Jace used Analogy 55% of the time when she solved open number sentences in the Individual Open Number Sentence Sessions (51 of the 93 open number sentences). Neither Alice, Kim, nor Jace used Analogy to pose stories in the Individual Context Sessions.

Algebraic Reasoning

The students did not appear to use Algebraic Reasoning in this study to pose stories for open number sentences. But, the students did use Algebraic Reasoning in this study to solve open number sentences. When using Algebraic Reasoning students in this study often changed the structure of the problem by creating a number sentence whose answer was mathematically equivalent. The students often referred to this as working "backwards." When the students referred to working "backwards," they were referring to applying the inverse of subtraction to a problem like □ – -2 = 1 by solving a mathematically equivalent to addition problem 1 + -2 = □. Alice demonstrated Algebraic Reasoning and "working backwards" in the following excerpt where she is solving □ – -1 =2 (see Figure 15). Although Alice also used Translation with counting, she drew upon Algebraic Reasoning by changing the structure of the problem first. Alice wrote 1 + -2 first. She stated, "I did one… I did it backwards. I did one plus negative two. And, I got negative one."

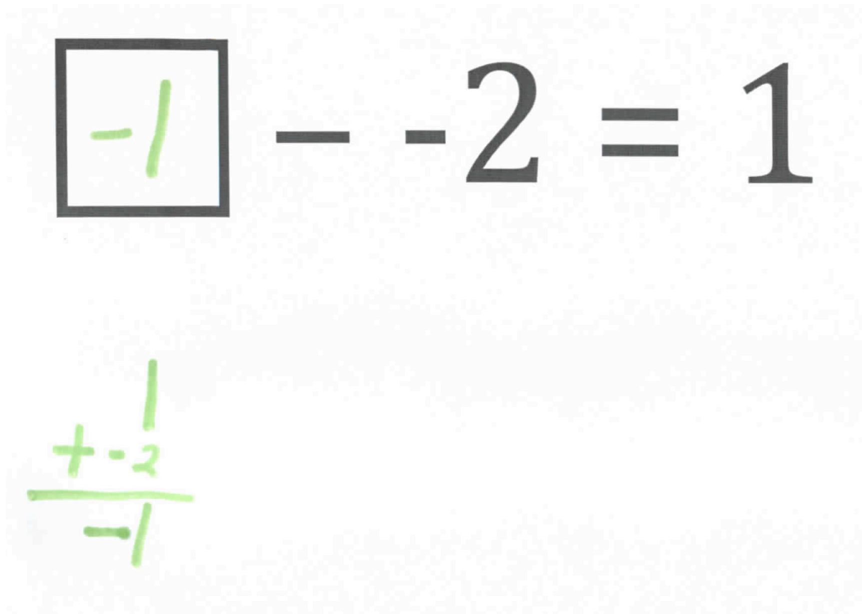


Figure 15. Algebraic Reasoning Open Number Sentence Example.

Algebraic Reasoning was also used in this study when students used algebraic properties, like the commutative property of addition. For example, Jace used Algebraic Reasoning when he solved -3 + □ = 14 when explicitly discussed the commutative property. Jace wrote 17 in the box and explained, "Because negative three is basically box (points at box) minus three. So, I did fourteen plus three and I got seventeen. And, seventeen minus three (points at -3) equals fourteen (points at fourteen). It's kind of like the commutative property." When asked what he meant by commutative property, Jace responded, "You just flip it around and you still get the same answer. Like fourteen minus three, I mean fourteen plus three is seventeen. And, seventeen minus three is fourteen."

Overall, Alice used Algebraic Reasoning 22% of the time when she solved open number sentences in the Individual Sessions (20 of the 93 open number sentences). Kim used Algebraic Reasoning 24% of the time when she solved open number sentences in the Individual Open Number Sentence Sessions (22 of the 93 open number sentences). Jace used Algebraic Reasoning 32% of the time when she solved open number sentences in the Individual Open Number Sentence Sessions (30 of the 93 open number sentences). Neither Alice, Kim, nor Jace used Analogy to pose stories in the Individual Context Sessions.

Multiple Uses of CMIAS

The students prevalently used multiple CMIAS when solving problems. Although the CMIAS were discussed in isolation above, the students often employed the use of multiple CMIAS when solving an open number sentence.

Translation and Proceduralization. In this example, Alice used Translation and Proceduralization when she solved $\square + -9 = -16$. To solve $\square + -9 = -16$, Alice used her fingers first to count. Then, she drew boxes and counted them and said "I think negative seven" and "I had nine and I added up until I got to sixteen." She used Translation when used the boxes, or quantities that she is keeping track of represent moving from -9 to -16. Alice then wrote -7 in the box, despite counting from 9 to 16 and having 7 boxes. Alice applied a Proceduralization that she could place -7 in the box without reference or justification to that rule or procedure. She calculated the sum of positive integers 7 and 9 without justification, "Because I did seven plus nine and I got sixteen." This was her use of Proceduralization because she treated this as an implicit rule that will always work.

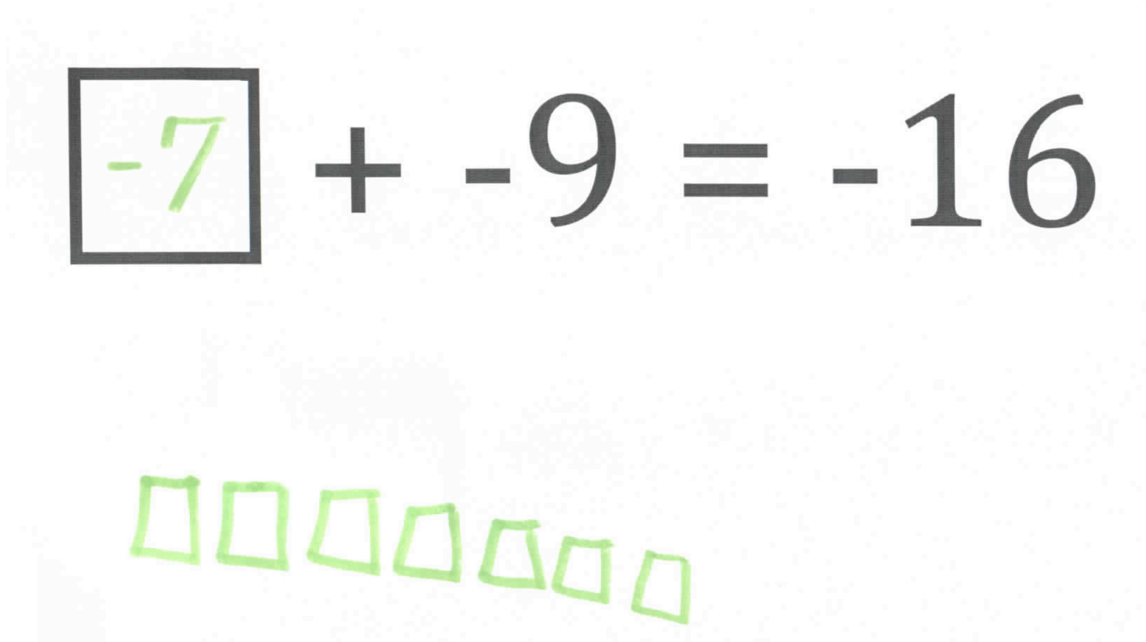


Figure 16. Translation and Proceduralization Open Number Sentence Example.

Algebraic Reasoning and Translation. In the following example Kim used Algebraic Reasoning and Translation when she solved 4 – □ = -12. Kim first wrote -16 in box. Using Algebraic Reasoning with a structure change of the problem, she stated, “Well, what I did to help me out here was I did (starts writing) twelve plus four and that gave me sixteen. I just didn’t use the negatives this time because it would just be easier to help me out.” Using Translation with a reference to movement, she explained, “Because then sixteen told me that that would be the box (points at -16) or like to make sure a negative number would let four get to negative twelve.”

Analogy and Counterbalance. In the following example Jace used Analogy and Counterbalance when he solved -20 + 15 = □. Using Analogy, Jace compared -20 + 15 to 20 – 15, “Because twenty minus fifteen equals five.” Then, using Counterbalance, Jace used the magnitudes of the two number, -20 and 15, to solve -20 + 15 = □, “But, it's negative twenty plus fifteen and the negative twenty is bigger than ... well, twenty is bigger than fifteen. And, twenty's the negative number. So you are still going to be in the negative numbers."

Summary of Overall CMIAS Use by the Students

Bookkeeping, Counterbalance, Relativity, Translation, Proceduralization, Analogy, and Algebraic were used by each of the students differently. Also, each of the CMIAS were used differently by the same students in the different sessions (i.e., Individual Context Sessions, Individual Number Sentence Sessions). Figure 17 illustrates the overall CMIAS use by the students with percentages.

	Alice		Kim		Jace	
	Context	Open Number Sentences	Context	Open Number Sentences	Context	Open Number Sentences
Bookkeeping	100%	20%*	75%	0%	71%	0%
Counterbalance	0	38	0	10	0	14
Relativity	0	0	0	0	11	0
Translation	0	39	25	54	18	25
Proceduralization	0	47	0	49	0	60
Analogy	0	24	0	61	0	55
Algebraic Reasoning	0	22	0	24	0	32

* Percentages do not add to 100% since more than one CMIAS could be assigned to unit of analysis.

Figure 17. Summary of Overall CMIAS Use by Students.

Looking across the columns representing the use of CMIAS in the Individual Context Sessions and the Individual Open Number Sentence Sessions, there is evidence that different CMIAS are utilized when posing stories and when solving open number sentences. For example, neither Kim nor Jace use Bookkeeping to solve the open number sentences; yet, Kim uses Bookkeeping 75% and Jace 71% of the time to pose stories for integer addition and subtraction problems. Alice only used Bookkeeping in the Individual Context Sessions; yet, Alice used Bookkeeping, Counterbalance, Translations, Proceduralization, Analogy, and Algebraic Reasoning in the Individual Open Number Sentence Sessions. Similarly, certain CMIAS like Proceduralization, Analogy, and Algebraic Reasoning were prevalently used by all three students while solving open number sentences, but not used when posing stories.

Within each of the cells, examining the percentages among the students, there is evidence that the students each draw upon different CMIAS and have different prominent CMIAS. For example, within the Individual Open Number Sentence Sessions, Alice generally used Bookkeeping and Counterbalance more than Jace and Kim. In these sessions, Kim's top three utilized CMIAS were: Analogy, Translation, and Proceduralization. Jace's top three utilized CMIAS were: Proceduralization, Analogy, and Algebraic Reasoning. And, Alice's top three utilized CMIAS were: Proceduralization, Translation, and Counterbalance.

Alice's Uses of the CMIAS During Individual Sessions

The first part of this chapter described CMIAS and the CMIAS that were used by the three Grade 5 students. Although important, the first part of this chapter only provided broad descriptions of the students' use of the CMIAS. The following section of this chapter provides an in-depth description of CMIAS use from one particular student, Alice. Alice was selected because Alice's mathematical thinking contrasted the most from Jace and Kim. Because Alice's CMIAS use was significantly different from Jace and Kim, and Jace is discussed in depth in Chapter V, Alice was selected. An in-depth portrait of Alice's CMIAS use is painted next.

Individual Context Sessions

During the all four of the Individual Context Sessions, Sessions 1–4, Alice used only Bookkeeping as she posed stories. Although she participated in the Group Sessions that made use of other contexts that may have promoted other CMIASs, she continued each session posing only Bookkeeping stories. For the most part, Alice used Bookkeeping to pose stories that used the integers as “wants,” “needs,” or “losses.” However, occasionally, although treating the integers in a counting manner and with a gain or loss to a singular quantity, didn’t apply ideas of opposites with the language “wants,” “needs,” or “losses.”

For example, Alice posed two different stories for $-12 + 7 = \square$. She posed, “I had negative twelve pencils and got seven more. Now I have negative five.” Here she used “negative twelve pencils” as an appropriate context. However, for her second story she posed, “I need twelve markers, or I need twelve pencils. I got seven. Now, I need five.” In this same session, she now used a “need” of twelve markers to represent the -12. She continued to use “needs” with the negative integers throughout this session. For example, for $-2 - 10 = \square$ she posed, “I had, I need two markers. My friend lost ten. Now, I need twelve markers.” She also used a “want” of an object to represent the negative integers. For example, she also posed for $-2 - 10 = \square$, “I want negative two pencils. I ... I lost ten. Now I need negative twelve.”

Sometimes Alice did not use the negative integers as a “loss” and the positive integers as a “gain.” Rather, she used the negative integers as a “gain” and the positive integers as a “loss.” For example, for $-15 + 4 = \square$, Alice posed the story, “I had fifteen pencils. I lost four. Now I have eleven pencils.” She used Bookkeeping in this story because a loss was applied to a singular quantity. Although conventionally and culturally we might not represent a quantity with a negative integer and addition as a loss, she uses Bookkeeping this way.

Individual Open Number Sentence Sessions

During the Individual Open Number Sentences Sessions, Alice used more of a variety of CMIAS than the Individual Context Sessions. She used Bookkeeping, Counterbalance, Translation, Proceduralization, Analogy, and Algebraic Reasoning. How Alice used each of these CMIAS during the individual open number sentence sessions is discussed next.

Bookkeeping. Alice used Bookkeeping in all four of the Individual Open Number Sentence Sessions, but had increased use of Bookkeeping in the second and third sessions. Alice used Bookkeeping in 2 of the 20 open numbers sentences or 10% of the time in Session 1 (see Figure 18). She used Bookkeeping in 6 of the 23 open number sentences or 26% of the time in Session 2. She used Bookkeeping in 8 of the 25 open number sentences or 32% of the time in Session 3. Alice used Bookkeeping in 3 of the 25 open number sentences or 12% of the time in Session 4.

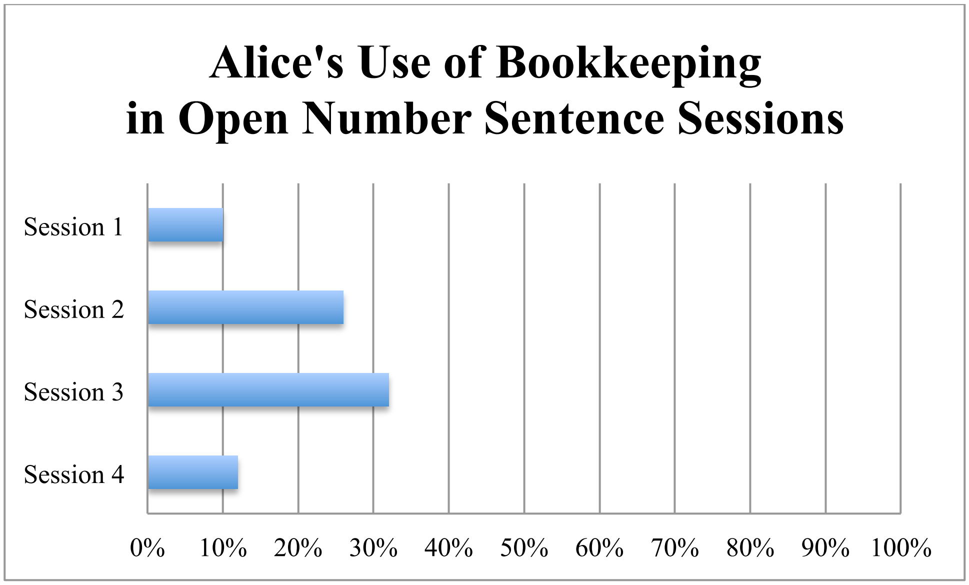


Figure 18. Alice's Use of Bookkeeping During Open Number Sentence Sessions.

When Alice used Bookkeeeping she treated the positive and negative integers as gains and losses. However, she often used the negative integers as the gains and the positive integers as the losses. For example, when Alice solved -18 + 12 = □ (see Figure 19). She first drew 18 tallies with green marker. Then, she used a pink marker and crossed off 12 of the tallies. Then, she wrote -6 in the box. She represented -18 with 18 tallies. She crossed off 12 of the tallies, which represented -12 to handle the addition of 12. She treated the addition of 12 as a loss of -12 by removing the 12 tallies, which represented -12.

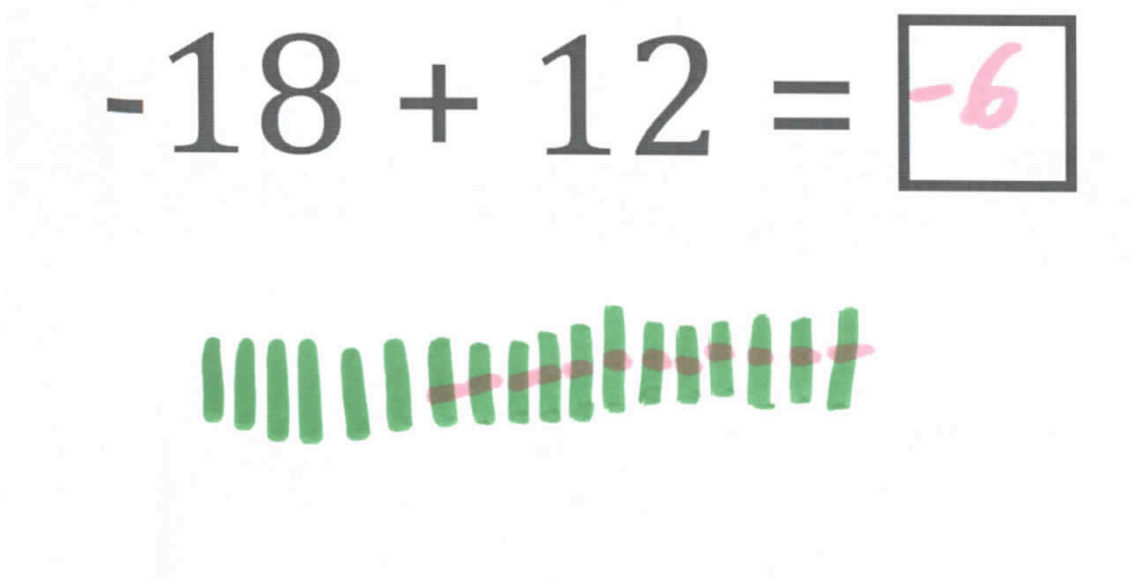


Figure 19. Alice's Drawing for Solving -18 + 12 = □.

Asked to explain her thinking for solving -18 + 12 = □, Alice shared, "I did eighteen lines with (points at the green tallies with finger), which that was the eighteen negative. I crossed off twelve (points at the right with the pink). And, I still had six negative left (points at the tallies that were on the left that had not been crossed off)." Alice represented the -18 with quantities of tallies and adding 12 as a loss of -12, or 12 tallies. Again, Alice used Bookkeeping when she solved -9 – 8 = □ in Session 2 (see Figure 20). Alice explained as she drew, "Because negative (draws 9 tally marks)...There's nine, nine, and then eight (crosses off eight tallies). And then there's one (circles one tally mark) negative left over." Here Alice represented -9 with 9 tallies. Then, to represent the subtraction of 8 she crossed off 8 tallies. However, those tallies actually represented -8. She actually drew -9 – -8 rather than -9 – 8 with this use of Bookkeeping.

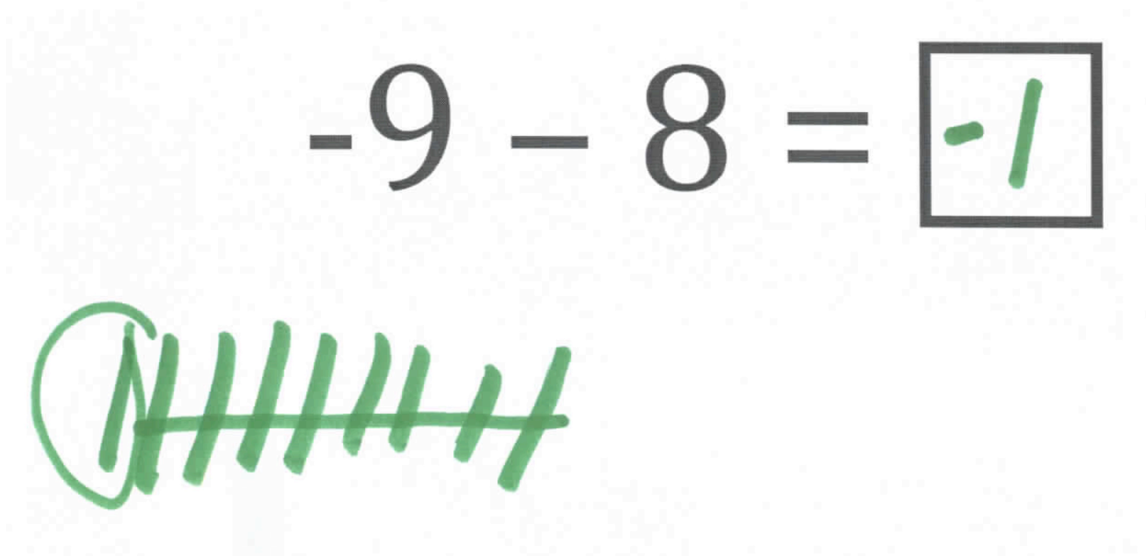


Figure 20. Alice's Drawing for Solving -9 – 8 = □.

Counterbalance. Alice used Counterbalance in all four Open Number Sentence Sessions, but Alice's use of Counterbalance mainly increased across the sessions. She used Counterbalance the most in Session 3 (see Figure 21). Alice used Counterbalance in 3 of the 20 open number sentences or 15% of the time in Session 1. She used Counterbalance in 9 of the 23 open number sentences or 39% of the time in Session 2. She used Counterbalance in 12 of the 25 open number sentences or 48% of the time in Session 3. She used Counterbalance in 11 of the 25 open number sentences or 44% of the time in Session 4.

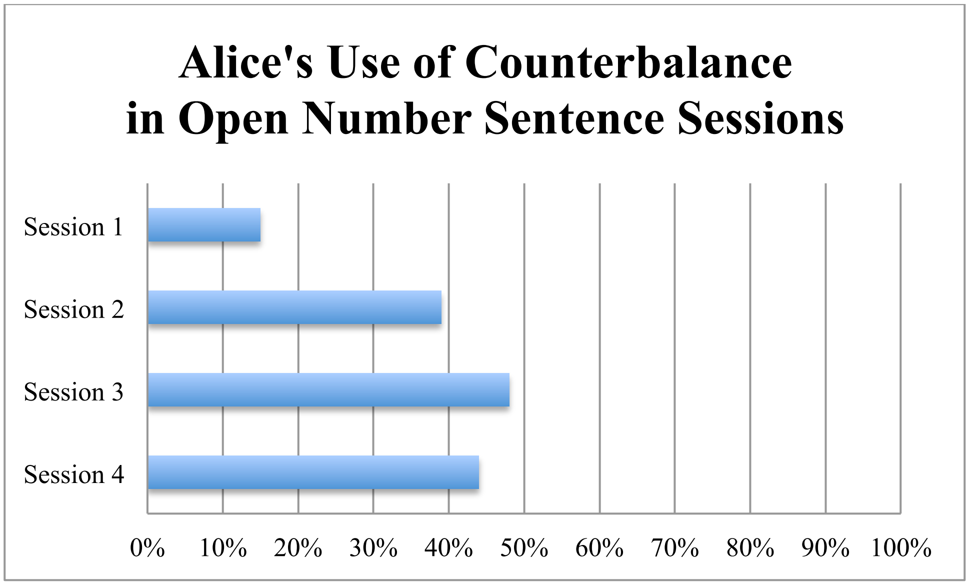


Figure 21. Alice's Use of Counterbalance During Open Number Sentence Sessions.

Alice used Counterbalance when she compared two different quantities. She often implicitly neutralized parts of the quantities and used the leftover quantity or remaining quantity as the solution. Alice often did this when subtracting two positive integers, and not just when adding or subtraction positive and negative integers. For example, Figures 21 and 22 illustrate the drawings Alice produced when using Counterbalance.

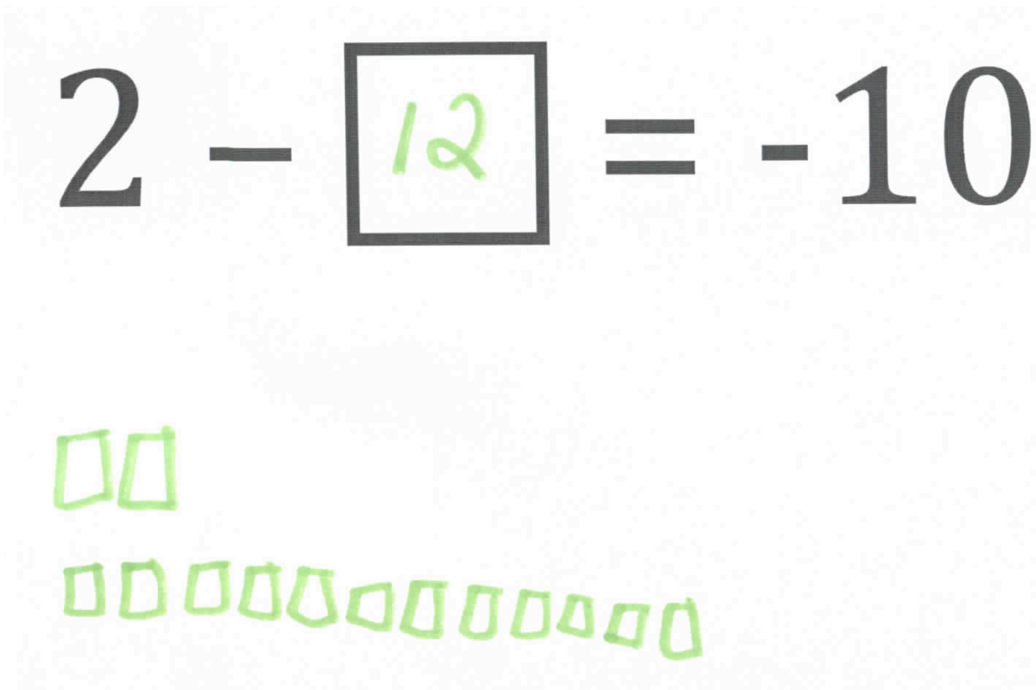


Figure 22. Alice’s Drawing for Solving 2 – □ = -10.

When Alice solved 2 – □ = -10 in Session 1 (see Figure 22) she used Translation and Counterbalance. She used Translation as she counted and Counterbalance as she compared the two different quantities:

A: (Draws two boxes, then moves pen to a lower position and draws more boxes. Uses fingers to count and then writes 12 in the box.)

T: Ok, can you explain what you did?

A: This is the two and then I count up until I got negative ten and I got twelve boxes.

T: Ok, so this is the two (points at upper two boxes). What was this one again (points at lower boxes)?

A: That was one, zero, negative one, negative two, negative three, negative four, negative five, negative six, negative seven, negative eight, negative nine, negative ten.

T: (pointing at each box) Ok. How did you get the twelve?

A: Because there are twelve boxes right here.

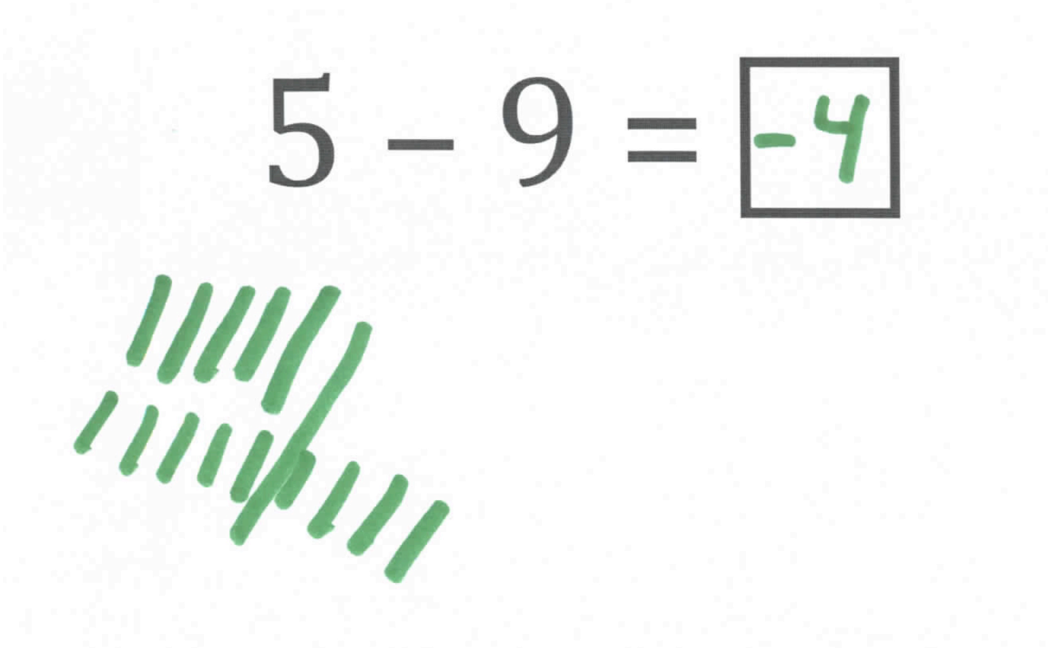


Figure 23. Alice's Drawing for Solving 5 – 9 = □.

Similarly, Alice used Counterbalance and Translation again in Session 2 when she solved 5 – 9 = □ (see Figure 23):

A: (Writes -4 in the box.)

T: Can you explain what you are thinking? You didn't write anything, you just did it in your head.

A: Well five is smaller than nine. And, I know that there's four left over and it's below zero and so I did four.

T: So how'd you know it was below zero?

A: Because I did five and I took nine away and I had four ... I did (begins to draw). I did five (draws five tally's), which I took nine away (draws nine more tally's below the first five tally's and aligns the tally's). Which that's that the whole numbers (points at the original five tally's), and so that means that there's four negatives. And so, I got negative (points at box).

This time Alice's used Counterbalance with her drawing and as she stated "Well five is smaller than nine. And, I know that there's four left over." Her drawing of the quantities, or tallies, came as part of her explanation at the end. Alice's acknowledgement that that -4 was "below zero" was considered use of Translation.

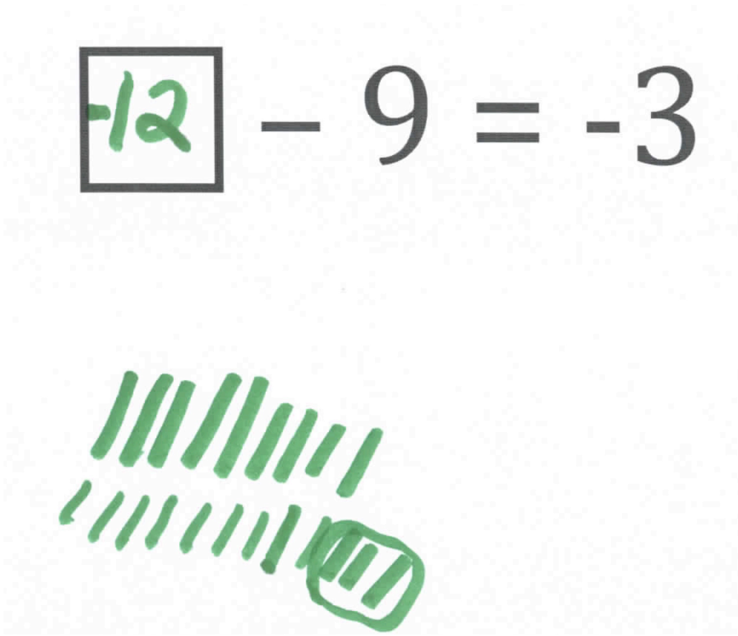


Figure 24. Alice's Drawing for Solving □ – 9 = -3.

Although Alice's use of Counterbalance and Translation together was productive, sometimes she encountered challenges with the use of both Counterbalance and Translation. For example, when Alice solved □ – 9 = -3 in Session 2 (see Figure 24), she incorrectly came up with the solution -12 using Counterbalance and Translation:

> (Draws tally marks/lines. Writes 12 in the box. Then adds a negative sign.) I did that because this is twelve (points to tally's) and this is nine. Well, negative twelve, you count. You go up nine and if ... from twelve you go up nine and there's these three (points at tally's) left over, which would be negative three.

Here Alice reasoned that she could compare -12 and 9 in a similar way as comparing 6 and 9 as she did above (see Figures 22 and 23).

Alice often used Counterbalance productively when solving addition problems with positive and negative integers (see Figures 22 and 23). In Session 3, Alice solved 15 + -24 = □ using Counterbalance (see Figure 25). She drew 15 tallies with green marker and drew 24 tallies with pink maker below the 15 green tallies. Then, Alice counted the tallies that did not have green tallies above them. She used the pink marker to count each of the tallies one by one and writes -9 in the box. Alice stated, “I did fifteen (points at the green tallies). And then twenty-four negatives (points at the pink tallies). And I had nine negative left.” And then she stated, “Because there was more, there was more not negatives. There was negatives left (uses right hand to point at the pink tallies that do not have green tallies above them.)”

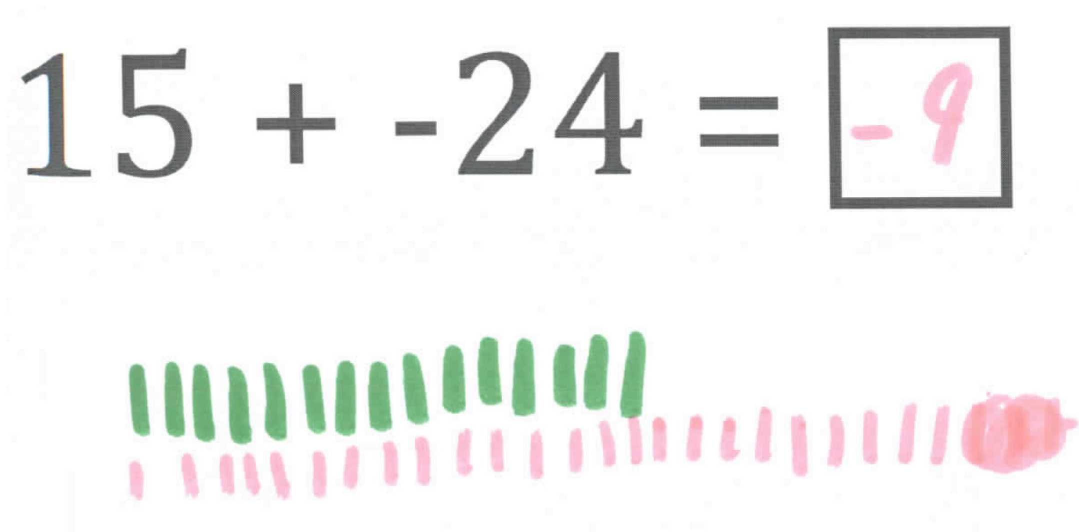


Figure 25. Alice's Drawing for Solving 15 + -24 = □.

Translation. Alice used Translation in all four of the Individual Open Number Sentence Sessions. Alice's use of Translation increased from Session 1 to 2, but then decreased in use in the remaining sessions (see Figure 26). Alice used Translation in 7 of the 20 open number sentences or 35% of the time in Session 1. She used Translation in 14 of the 23 open number sentences or 61% of the time in Session 2. She used Translation in 8 of the 25 open number sentences or 32% of the time in Session 3. She used Translation in 7 of the 25 open number sentences or 28% of the time in Session 4.

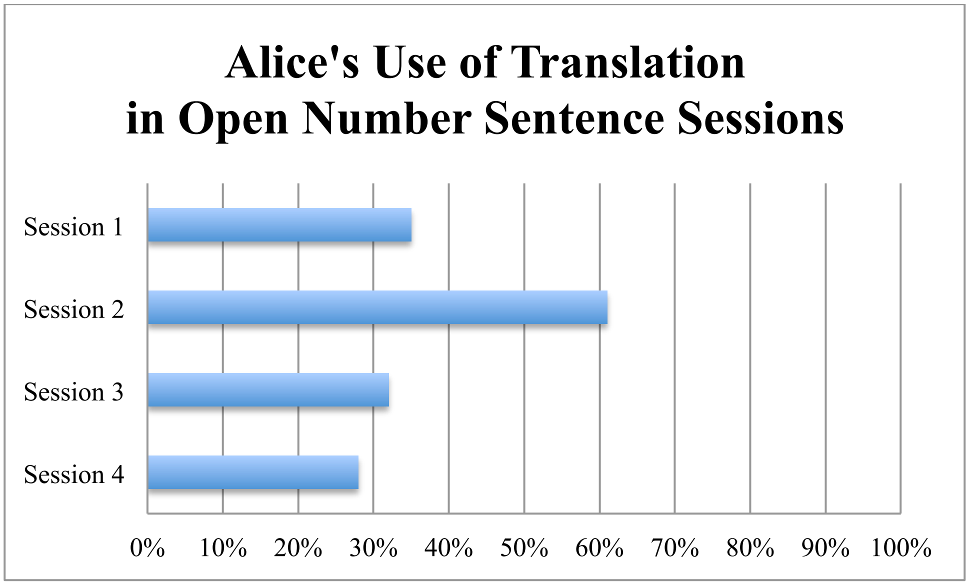


Figure 26. Alice's Use of Translation during Open Number Sentence Sessions.

Alice used Translation to solve -1 – □ = 8 in Session 1. Alice drew a picture where she used boxes to count from -1 to 8 (see Figure 27) and shared:

(Drew boxes, counted boxes, wrote 9 in the box.) I got this because this is a negative one (points at leftmost box). This is zero (points at the next box)and then I counted up to eight (runs finger along the rest of the boxes) and I got nine.

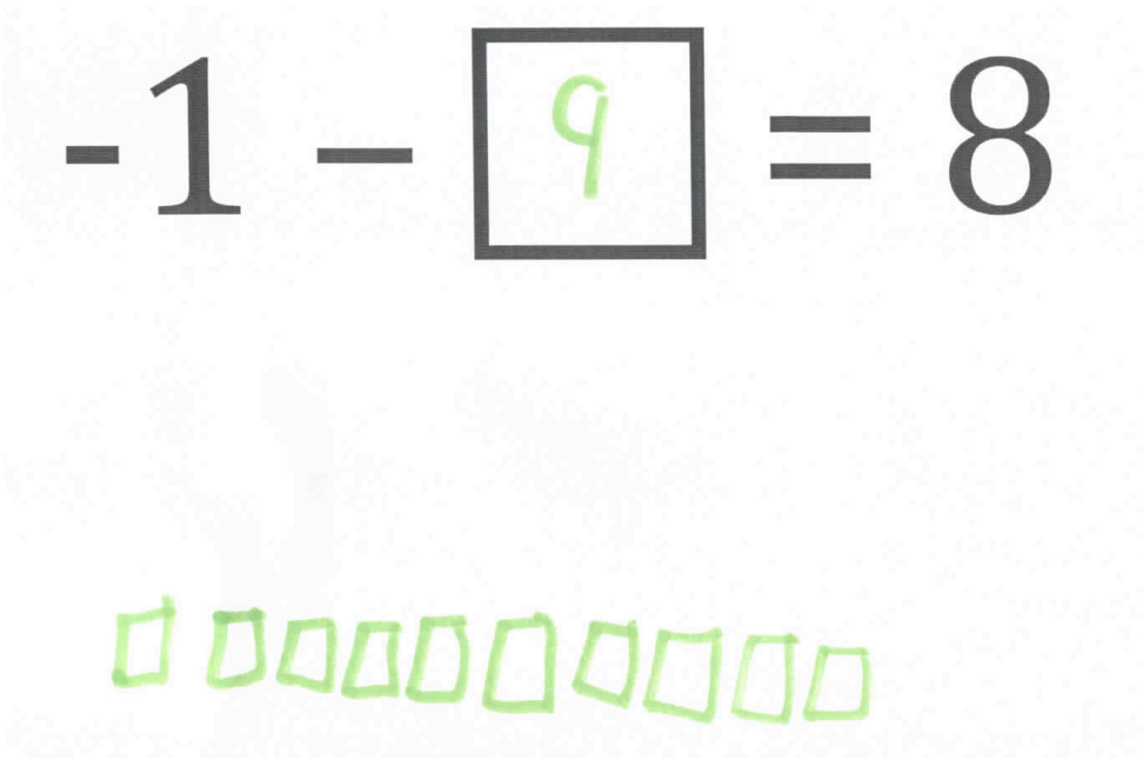


Figure 27. Alice's Drawing for Solving -1 – □ = 8.

Although Alice did not get the correct answer, she used Translation when she treated each of the boxes as a position (e.g., -1, 0, 1) and used the boxes to record her counting from -1 to 8, which were 9 boxes.

Alice used Translation when she solved -4 + □ = -19 in Session 2. She first drew the tallies that were layered (see Figure 28) and then stated, "I did four and then I added a line till I got to nineteen. And, then I got fifteen, negative fifteen plus four to check my work. And, I got nineteen." Alice's use of Translation was reflected in her use of the tallies that she drew to count from -4 to -19.

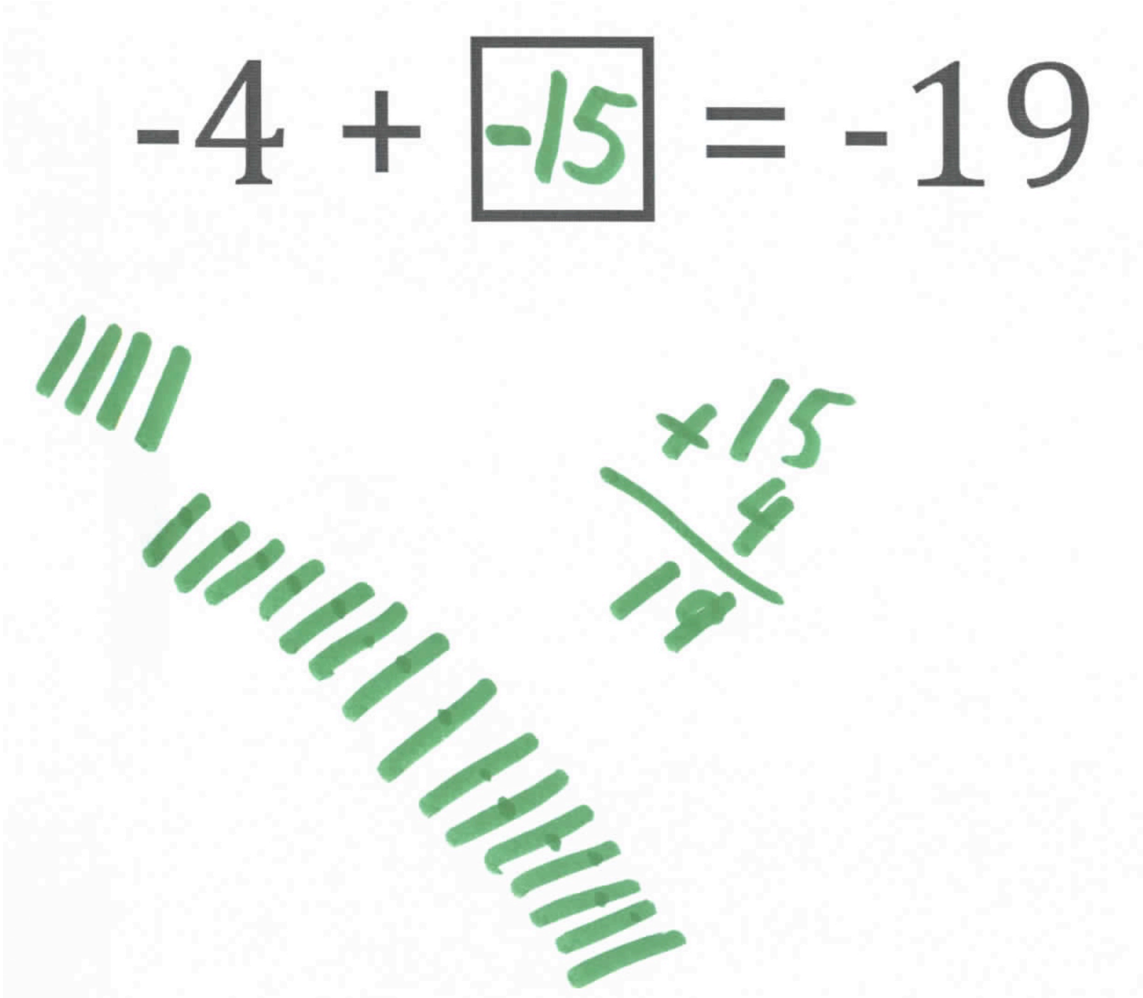


Figure 28. Alice’s Drawing for Solving -4 + □ = -19.

Alice’s drawing for solving -4 + □ = -19 began with representing -4 with 4 tallies and then she created a lower row of tallies and continued counting until she got to 19 tallies. She then counted the lower row of tallies.

By Session 3 and 4, Alice only utilized Translation alongside other CMIAS. For example, in Session 4 Alice used Translation and Proceduralization to solve -4 + □ = 10. To solve -4 + □ = 10 she drew a picture (see Figure 29), wrote 14 as the answer in the box, and explained:

Because if there's four, four negatives (draws more tallies) and there's fourteen (positives), you take four off because you are going up. Then you can past. So it would just be like subtracting, if it's negative plus positive and there's ten so I knew it that it would be fourteen.

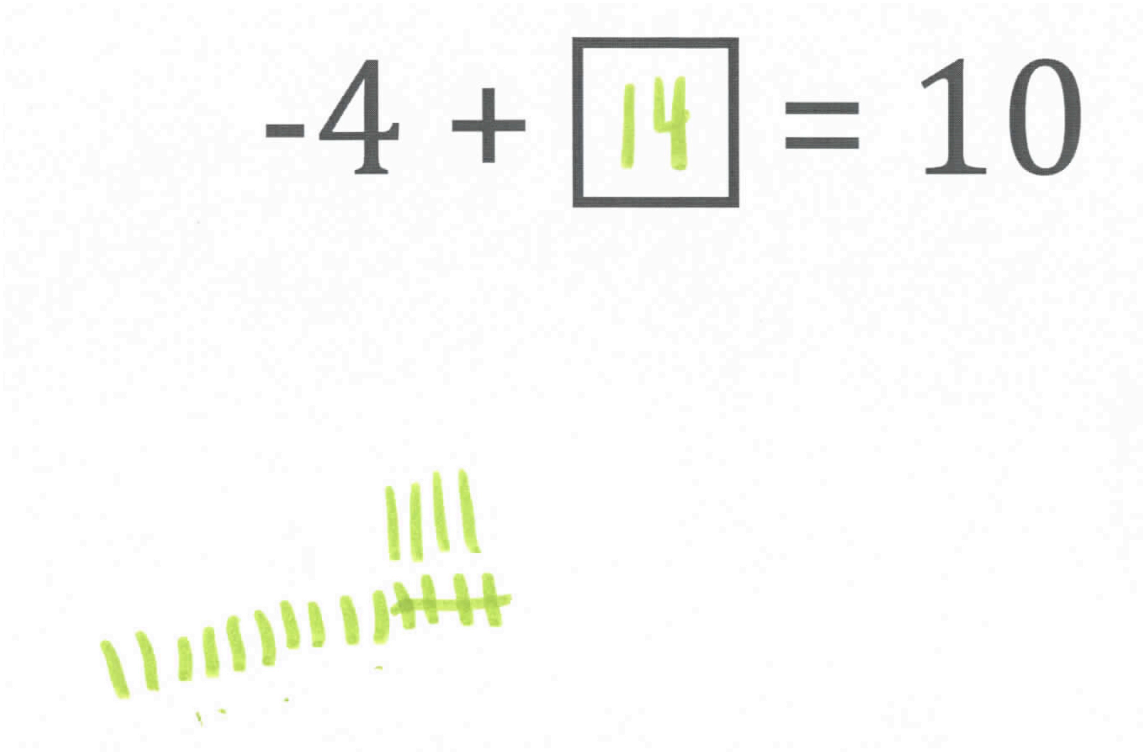


Figure 29. Alice's Drawing for Solving -4 + □ = 10.

Alice used Translation as she reasoned that "you are going up" and "then you can past." She also used Proceduralization when she described her rule and began to generalize her rule for "if it's negative plus positive."

It is noteworthy to point out that Alice did not draw a conventional number line or empty number line when she used Translation in any of the sessions. Rather, Alice often applied Translation without a drawing or to her drawings that utilized tallies. Also, Alice used Translation subtly and paired with other CMIAS, like Counterbalance. For example, she used Counterbalance and Translation as she solved 2 – □ = -10 in Session 4 (see Figure 30).

A: Well, I did two (starts to draw tallies) ... Wait. Actually it's negative twelve (adds a negative sign to the 12).

T: Ok.

A: I did two (points the tallies) and this is the positive. Then (draws more tallies to the right and below the original tallies). And then I did negative (points to the tallies to the right) these are the negatives. And, I'm going to cross off the positives (crosses off tallies). And then there's ten left.

T: Ok. So, so, can you explain your picture (points to the tallies) to me one more time? Sorry, I was just trying to get it.

A: These (points at two tallies) are the two positives. And then, these are the twelve negatives points to the tallies to the right). And well, I crossed off. There's two positives. So I added, if you add two you are going up in the negatives. So, it's ... there's ten left in the negative ten.

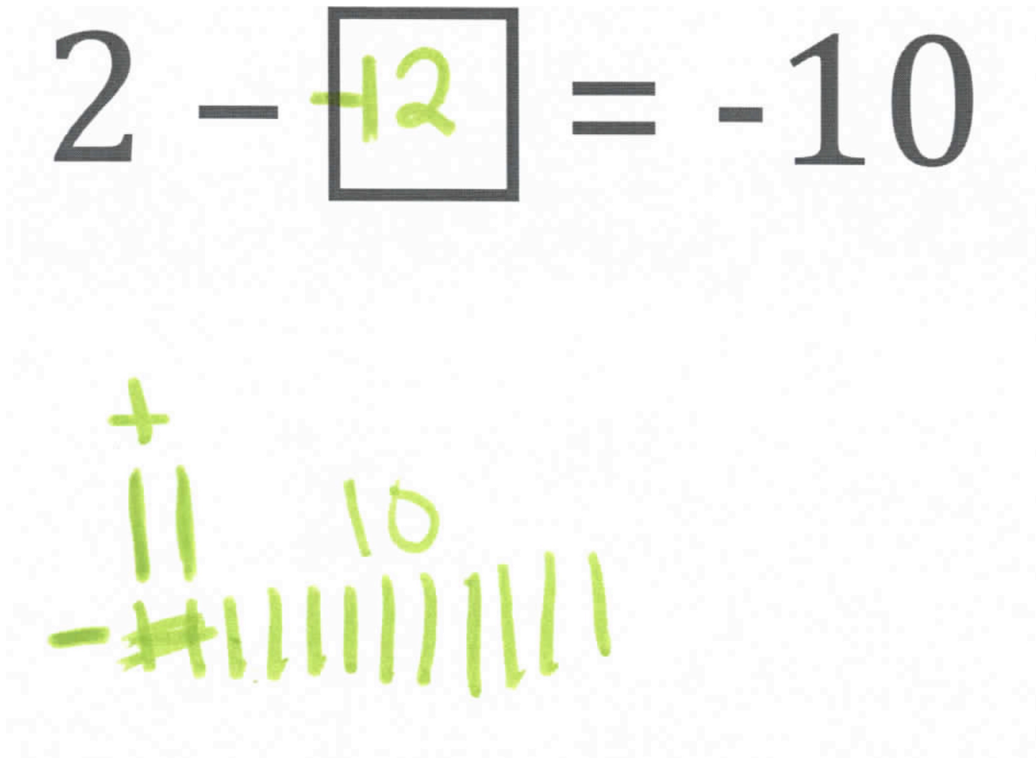


Figure 30. Alice's Drawing for Solving 2 – □ = -10.

Alice's use of Translation was supported by her statement that "you are going up in the negatives." Alice used Counterbalance as she compared and reasoned between two different quantities, "These (points at two tallies) are the two positives. And then, these are the twelve negatives points to the tallies to the right)."

Proceduralization. Alice used Proceduralization in all four of the Individual Open Number Sentence Sessions. Alice's use of Proceduralization increased from Session 1 to 2, decreased from Sessions 2 to 3, and then increased form Sessions 3 to 4 (see Figure 31). Alice used Proceduralization in 6 of the 20 open number sentences or 30% of the time in Session 1. She used Proceduralization in 10 of the 23 open number sentences or 43% of the time in Session 2. She used Proceduralization in 10 of the 25 open number sentences or 38% of the time in Session 3. She used Proceduralization in 18 of the 25 open number sentences or 72% of the time in Session 4.

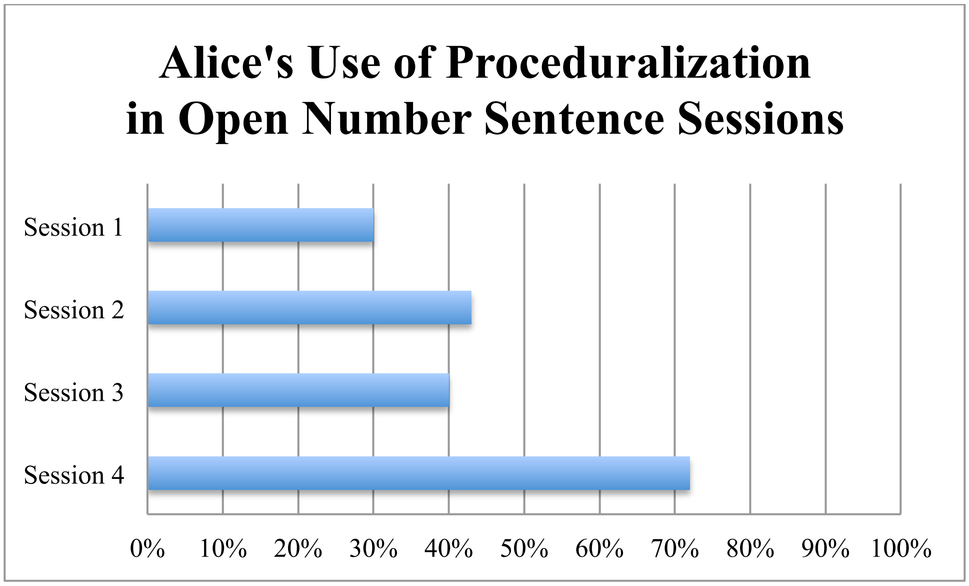


Figure 31. Alice's Use of Proceduralization During Open Number Sentence Sessions.

Alice's explanations sometimes focused on the steps she took to solve the problems, when this happened it was considered Proceduralization. Alice used Proceduralization, without another CMIAS, only once in Session 1 when she solved -8 + - 7 = □:

A: (Writes -8 + -7 vertically). Negative eight plus negative seven and I got fifteen (writes negative fifteen).

T: Can you explain what you are thinking?

A: I did negative eight plus negative seven because that's how I have them. So I did eight plus seven (points to vertical drawing) and got fifteen and then I did I did a negative.

Here Alice's focus was on the steps she took to solve the problem. Alice used only Proceduralization, without another CMAIS, also only once in Session 2 when she solved □ – -1 = 4 (see Figure 32):

(Writes 5 in the box.) Five minus one is ... (Hesitates and places marker by five. Then draws tally marks/lines.) Five minus one is four. So I did five (points at tally marks.)

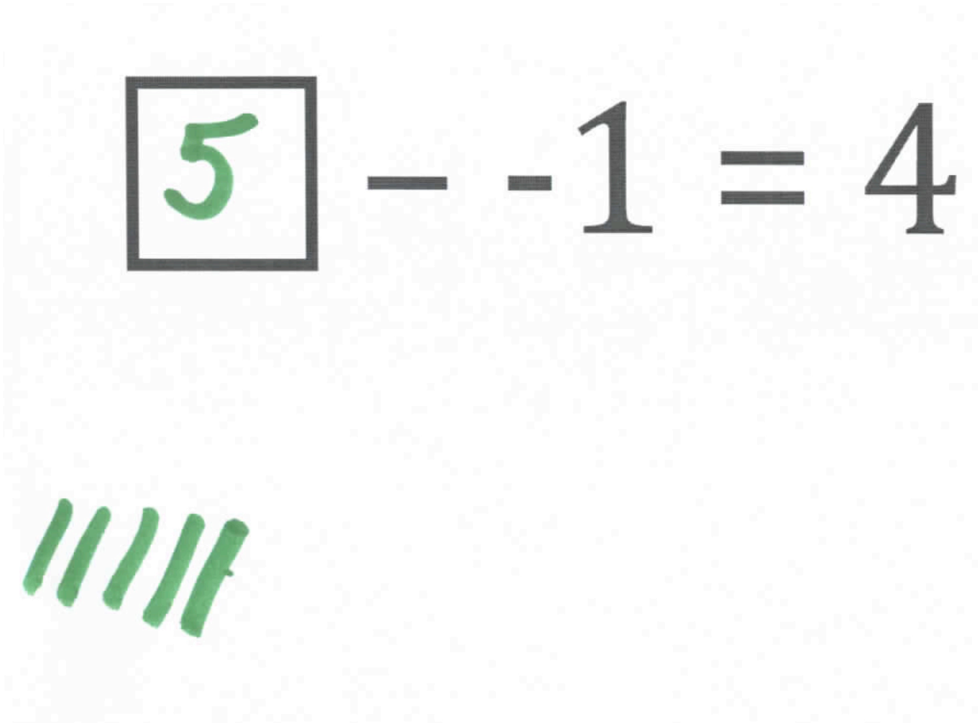


Figure 32. Alice's Drawing for Solving □ – -1 = 4.

Again, Alice used Proceduralization as she described her steps, or her procedure for solving solving □ – -1 = 4. Alice, in Session 3, as she solved -17 + -6 = □ stated, "Well, if it's negative plus negative equals a negative. So, I just did it like a normal problem." This generalization of "if it's a negative plus negative equals a negative" was considered use of Proceduralization. This was the only occurrence of just Proceduralization, without another CMIAS, in Session 3. In Session 4, Alice used only Proceduralization three times. These uses were while she solved the open number sentences: -8 + -7 = □, 2 – -3 = □, and □ – -3 = 0. The following transcript illustrates how Alice solved 2 – -3 = □ using Proceduralization (see Figure 33).

A: (Writes 5 in the box.)

T: Ok. How'd you get five?

A: Because it's just like the last one. You do plus (changes minus sign to plus sign) and take that off (scratches off the negative symbol of -3). And, it just be like two plus three.

T: Ok. So, how could you explain that to somebody that maybe ... like the rest of your class hasn't had the experiences that you guys had. So, how might you convince somebody that that's true?

A: Well, they would think it's two because they don't know the right answer.

T: (laughs) But, what if they were like I don't believe you. How might you try to convince them?

A: I'll say that... I'll show how I figured it out. Like a two (draws two tallies). I don't know (laughs).

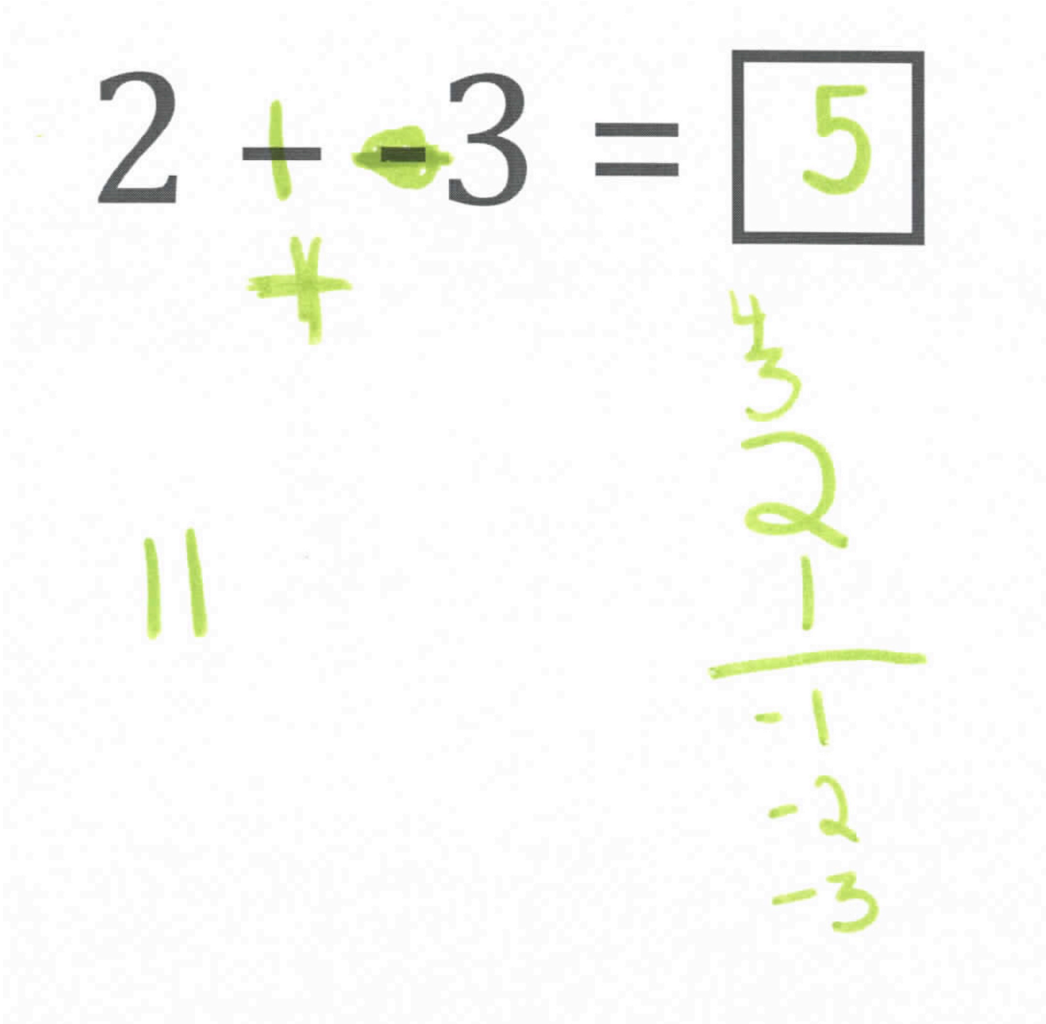


Figure 33. Alice's Drawing for Solving 2 – -3 = □.

Although Alice was able to solve the open number sentence 2 – -3 = □ correctly, she was not able to explain why it worked or draw a picture that supported her reasoning. Alice tried drawing both tallies and a number path/number line to explain her reasoning (see Figure 32) and was unable to come up with a way to explain 2 – -3 = □ other than using Proceduralization. Alice mostly used Proceduralization with other CMIASs in all of the sessions. For example, Alice used Counterbalance and Proceduralization when she solved -7 + □ = -2 in Session 4:

> Well, I knew that two (points at -2). I knew that seven (points at -7) minus (points at plus sign) five (points at five) is two. And seven, it's bigger than five, so it'd be negatives. So it would be negative two. And, then it would, and five would be in the box because it equals negative two.

Alice used Counterbalance as she made a magnitude comparison of -2 and -7. She also used Proceduralization as she described her rule or procedure for solving this problem. Alice also used Proceduralization and Translation when she solved 1 – □ = 3 in Session 4. Alice used Proceduralization when she drew upon a rule she developed, “Because it's, since it's negative plus not ... I forgot what's it's called... positive. I forgot what it was called for a minute. Since it was negative plus positive that it'd be like adding instead of subtracting, so one plus two it equals three.” Alice again explained her procedure, “This is a positive (points at 1) and this is a negative (points at -2) and so if you that (scratches off the negative symbol and changes the minus symbol into a plus symbol) you could do one plus negative two equals three.” Alice initiated another discussion about this particular open number sentence after solving another several other open number sentences. It was during this discussion that she thought the answer was not -2 as she attempted to reason with Translation (see Figure 34):

> I'm going to make a number line (draws a line in the lower left corner). So here's my one, two. So this is where I'm at (puts a maker next to the one on the number line). And, then you need to subtract two. One ... (draws on number line). It would be negative two. Negative two would be the answer. So, it wouldn't be that. It wouldn't be two (points at the problem 1 + -2 = 3), negative two.

As the session continued, Alice reflected on her how she was unsure about the answer, "I wrote negative two. Then, I crossed it off. Then, I wrote negative two and I crossed it off. Then, I wrote negative two again." Alice again used Translation to reason that the answer was -2.

A: (laughs) What can you add… what can we subtract to get to three. That's confusing. Well, if it was negative...(smiles, laughs). We need to get up here somewhere when we are subtracting. (laughs)

T: (laughs) Why is that making you laugh?

A: Because the ...we're subtracting, but we are trying to get to a higher number. So, this (points to the box in $1 - \square = 3$) has to be negative, I think. Yeah, I think this has to be negative. (Lays head down.) Because, I keep going back... I think it's negative two because ...(laughs)

T: (laughs)

A: Negative two again. (laughs)

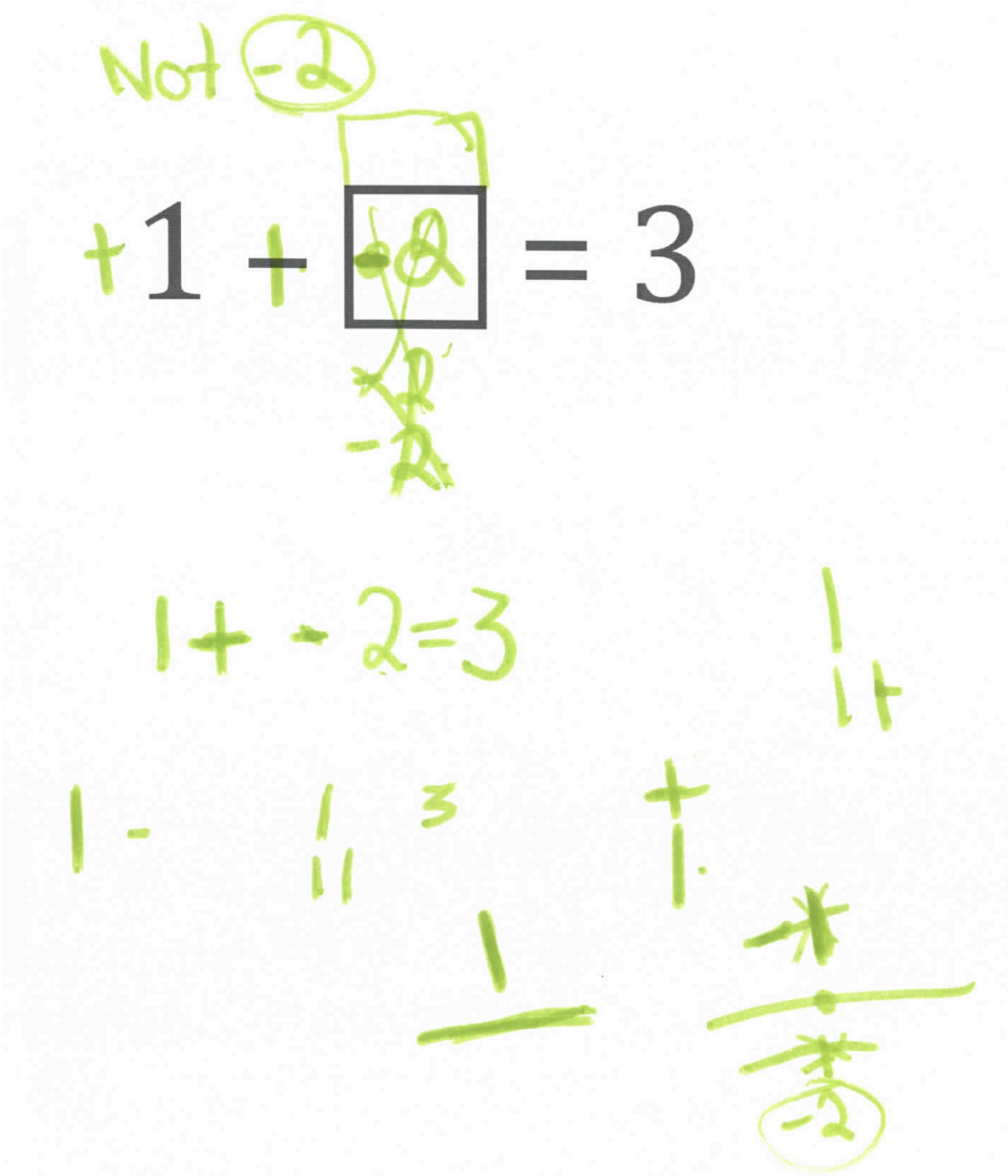


Figure 34. Alice's Drawing for Solving 1 – □ = 3.

Alice's use of Translation was verbalized when she stated, "We need to get up here somewhere," and "we're trying to get to a higher number." As illustrated by Alice's transcript and drawing, she eventually determined the correct answer using a coordination of both Proceduralization and Translation.

Analogy. Alice used Analogy in all four Individual Open Number Setence Sessions. Alice's use of Analogy was nearly the same across all four sessions (see Figure 35). Alice used Analogy in 5 of the 20 open number sentences or 25% of the time in Session 1. She used Analogy in 5 of the 23 open number sentences or 22% of the time in Session 2. She used Analogy in 6 of the 25 open number sentences or 24% of the time in Session 3. She used Analogy in 6 of the 25 open number sentences or 24% of the time in Session 4.

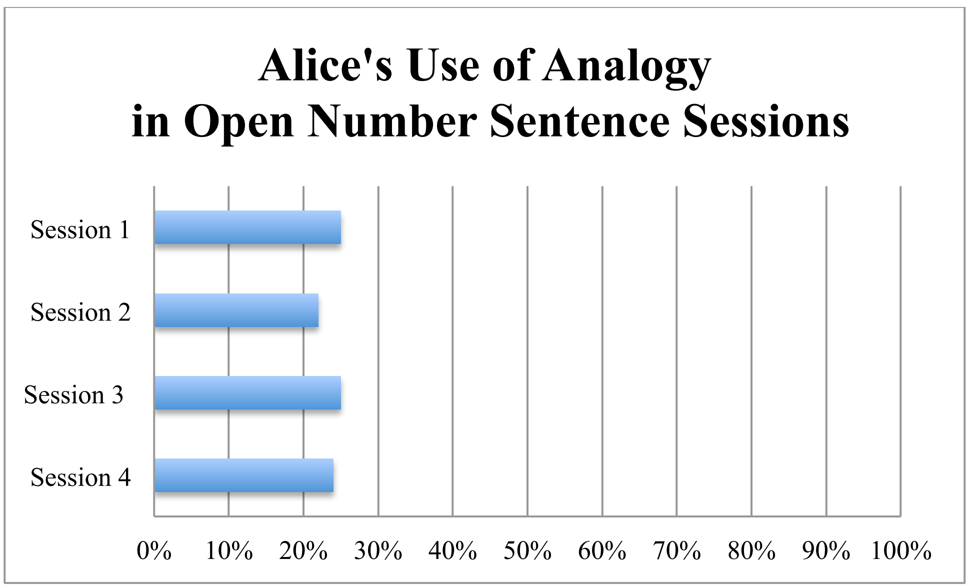


Figure 35. Alice's Use of Analogy During Open Number Sentence Sessions.

Alice nearly always used Analogy paired with other CMIASs, with one exception. Alice used only Analogy in Session 3 when she solved □ – -5 = 0. The following transcript excerpt illustrates how Alice connected 5 – 5 to -5 – -5:

A: (Writes -5 in the box.)

T: Ok. How'd you know it was negative five?

A: Because negative five minus (starts to draw vertical problem) negative five that would be zero.

T: But, why?

A: Because if you have five and you got rid of five, that would mean that you had zero left.

Alice directly compared 5 – 5 to -5 – -5, but this was the only open number sentence that used Analogy alone.

All of the other uses of Analogy, Alice used Analogy alongside other CMAISs. The following transcript excerpt highlights Alice's use of Bookkeeping and Analogy when she solved □ + -3 = 7 in Session 1 (see Figure 36).

A: (Writes 7 – -3 vertically, then draws seven boxes and crosses off three boxes. Goes back to vertical number sentence drawing and then writes -5 in the box.) I drew seven boxes and then I minused three (counts boxes and motions with marker, changes -5 to -4).

T: Ok.

A: And, I got four, so then I did negative four.

T: What made you do negative four?

A: Because well four plus three would give you seven and negative, if you had ... and then I added the negative sign because four, negative four plus negative three would be seven.

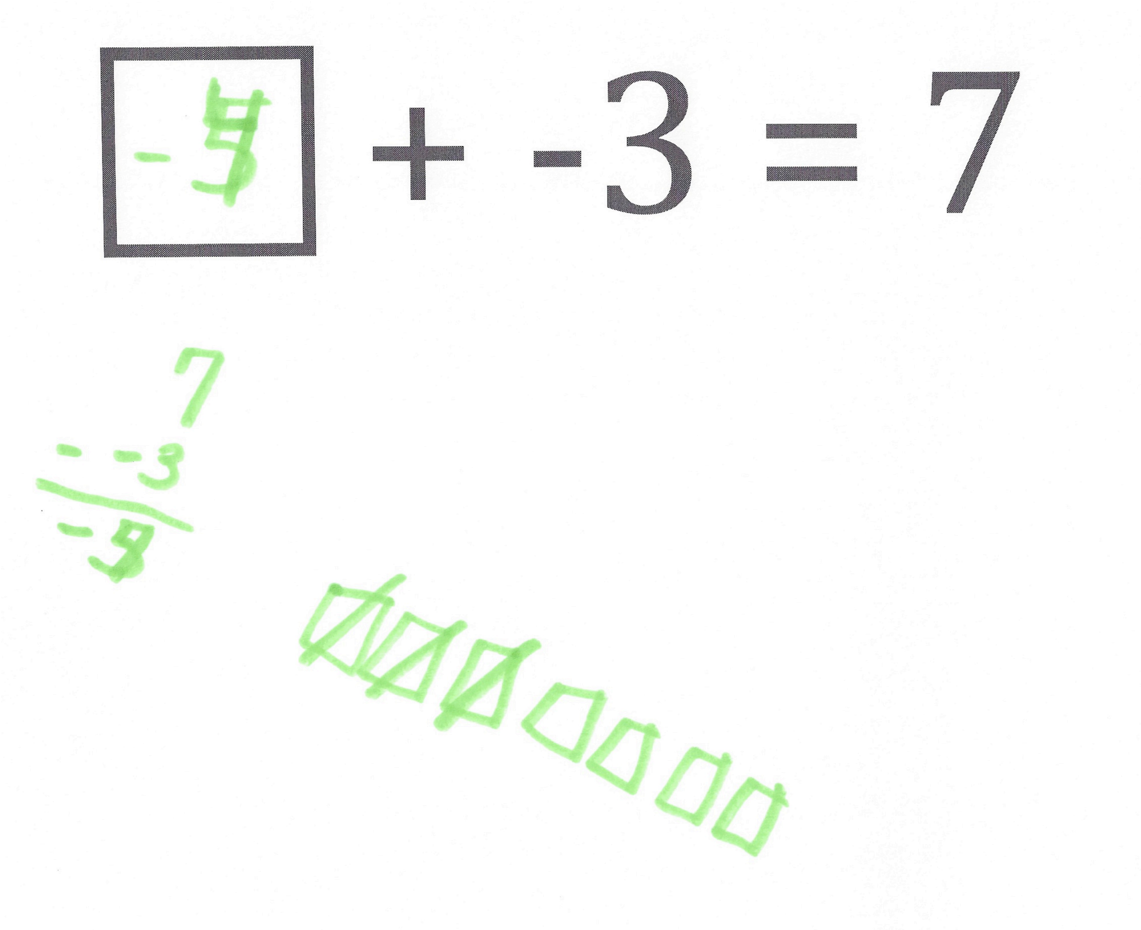


Figure 36. Alice's Drawing for Solving □ + -3 = 7.

When solving □ + -3 = 7, Alice used Bookkeeping when she treated addition of -3 as a loss, when she crossed the boxes off of her singular quantity of seven boxes. She used Analogy when she compared this problem to 4 + 3 when she stated, "Because well four plus three would give you seven and negative."

Alice also used Bookkeeping and Analogy together in in Session 2 when she solved □ + 19 = -4 (see Figure 37). The following transcript excerpt highlights her use of Bookkeeping as she used tallies to apply gains and losses of a singular quantity. Alice used Bookkeeping to solve □ + 19 = -4 when she treated the -4 as a loss of 4 applied to the quantity of 19. And, she used Analogy when she compared -15 + 19 to 15 + 19, “If it was just fifteen, it would be like thirty.”

A: (Draws tallies. Crosses tallies off. Counts tallies. Writes -15 in the box.)

T: Ok. Can you explain what you are thinking?

A: I did nineteen. Then, I crossed off four. And I counted how many I have left, which I got fifteen. Then I did nineteen minus negative fifteen and I got four.

T: How'd you know it was negative fifteen?

A: If it was just fifteen, it would be like thirty.

T: Ok. When you first started doing it, you did these ones (points at lines), and then you drew these ones (points at the other lines), and then you drew these ones. Can you tell me what you are thinking there?

A: I just did that to remember that I have to cross off four.

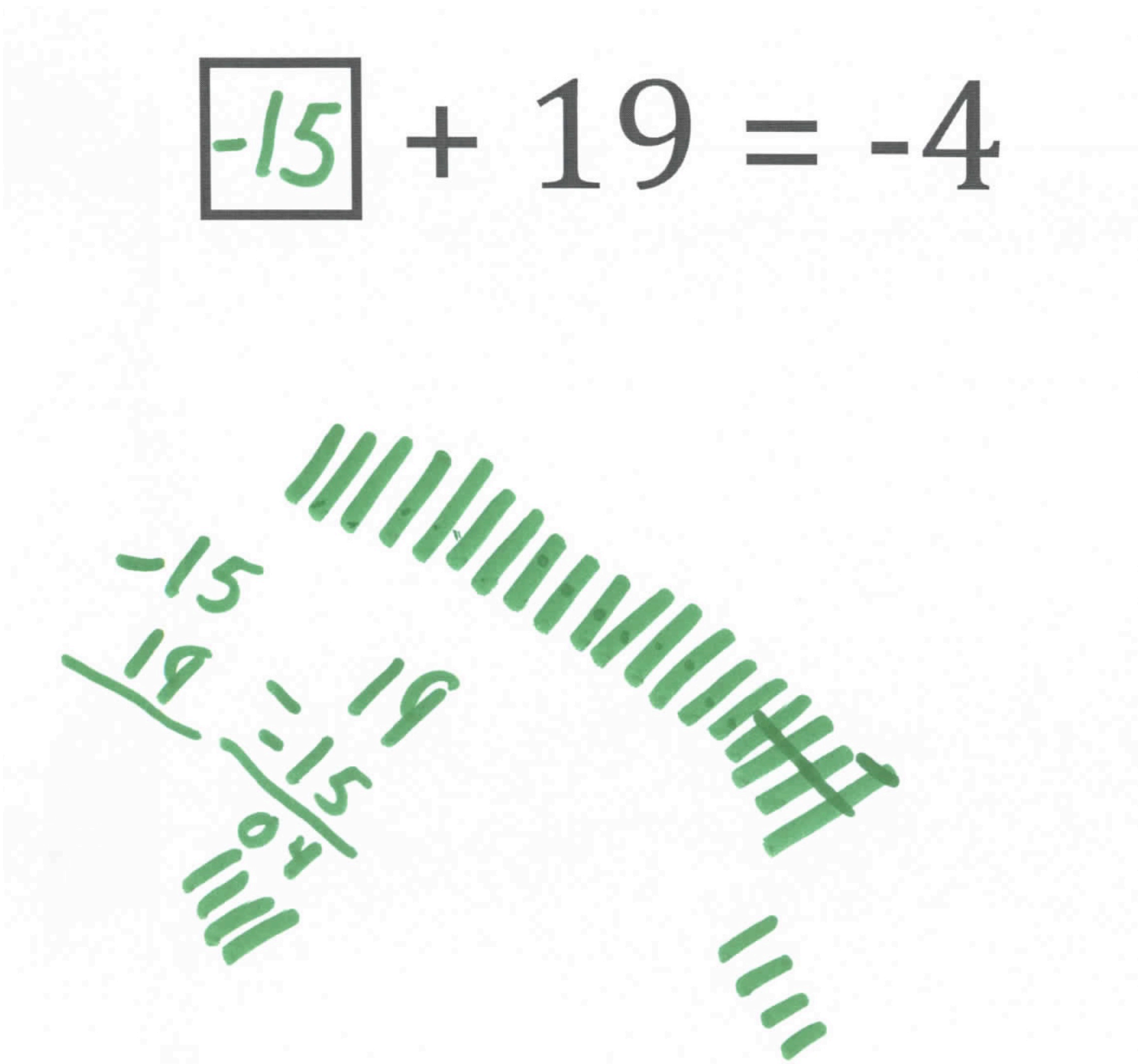


Figure 37. Alice’s Drawing for Solving □ + 19 = -4.

Although Alice used Bookkeeping and Analogy in ways that produced the incorrect answers in Sessions 1 and Session 2, Alice used Bookkeeping and Analogy in Session 3 to get the correct answer when she solved -9 + □ = -3 (see Figure 38).

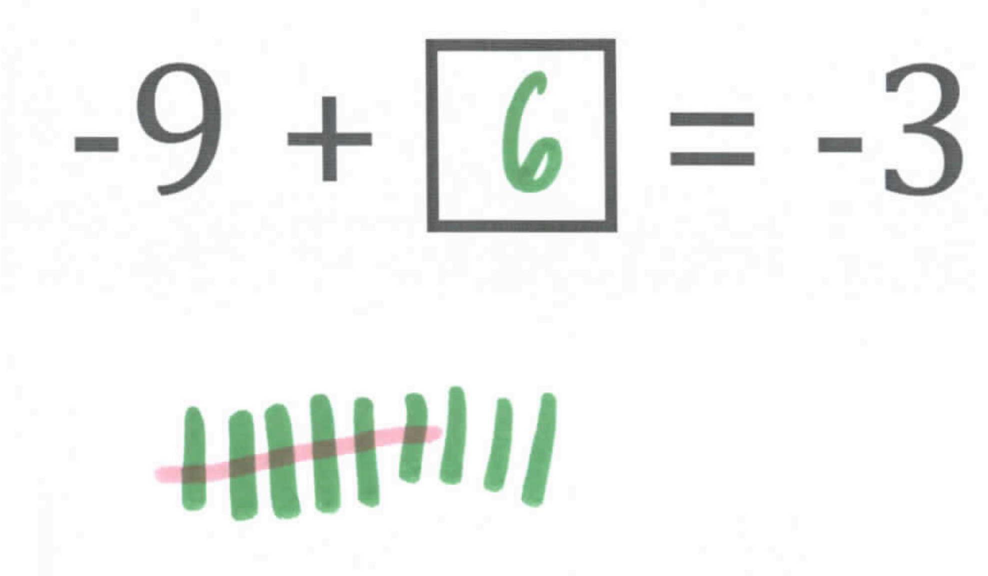


Figure 38. Alice’s Drawing for Solving -9 + □ = -3.

She used Analogy when she utilized 3 + 6 to solve the problem and used Bookkeeping when she treated adding 6 as a loss of six tallies:

> Because I knew that three plus six was nine. So if you ... I did this in my head (Draws nine tallies.) And then I knew these are the negatives. I crossed off (uses pink marker to cross off six tallies). Then I would have three negatives left.

Alice used Analogy with other CMIASs as well. For example, Alice used Analogy and Proceduralization together when she solved -1 – □ = 8 in Session 4 (see Figure 39). She used Analogy when she compared -1 – □ = 8 to another number sentence, “It would be negative eight if it was (draws horizontal problem -1 – -9 = -8 on paper).” And, she used Proceduralization when she described a rule, “Because it would be like the last time. Where I did, I took that off (scratches off negative symbol of -7) and I turn this into adding (changes minus sign to plus sign). One plus seven equals eight.”

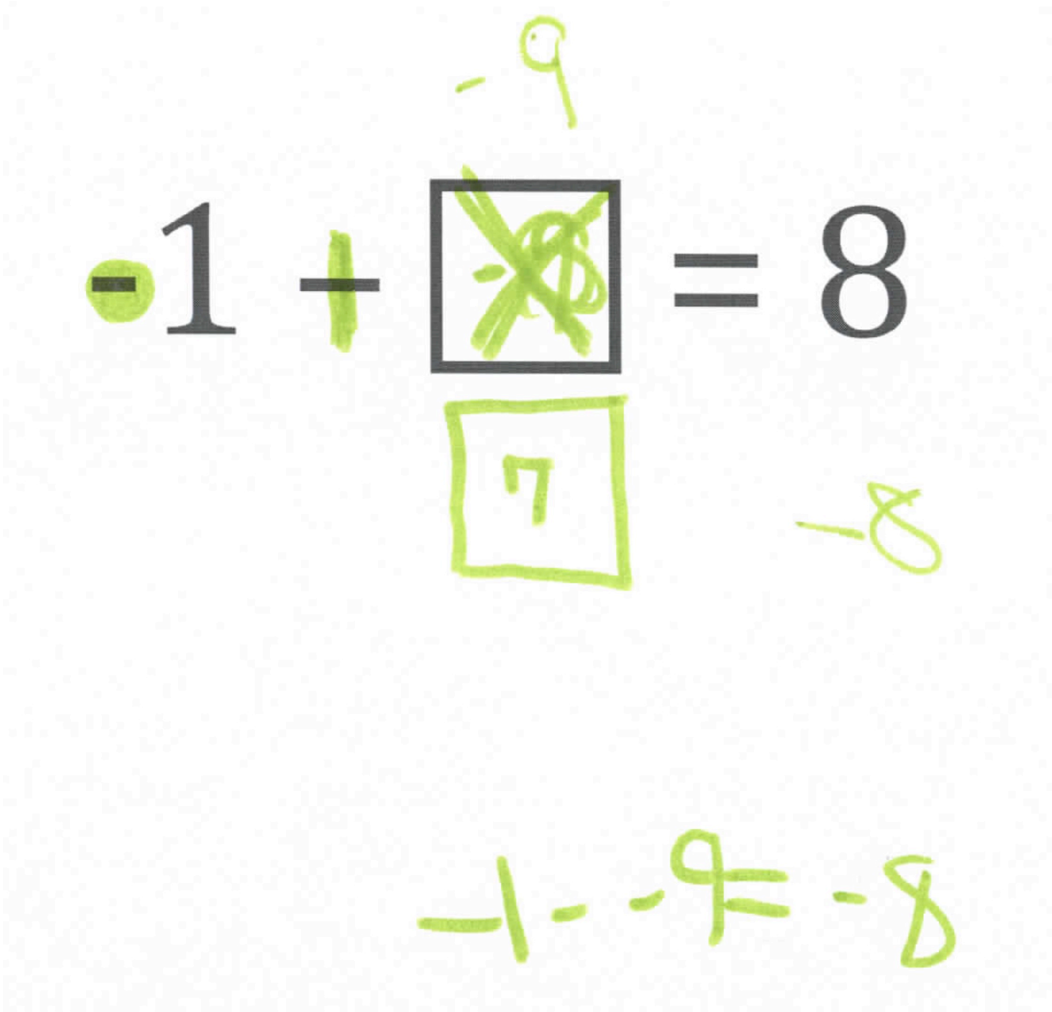


Figure 39. Alice's Drawing for Solving -1 – □ = 8.

Algebraic Reasoning. Alice used Algebraic Reasoning in all four Individual Open Number Sentence Sessions. Alice's use of Algebraic Reasoning declined in use across all four sessions (see Figure 40). Alice used Algebraic Reasoning in 10 of the 20 open number sentences or 50% of the time in Session 1. She used Algebraic Reasoning in 4 of the 23 open number sentences or 17% of the time in Session 2. She used Algebraic Reasoning in 3 of the 25 open number sentences or 13% of the time in Session 3. She used Algebraic Reasoning in 3 of the 25 open number sentences or 12% of the time in Session 4.

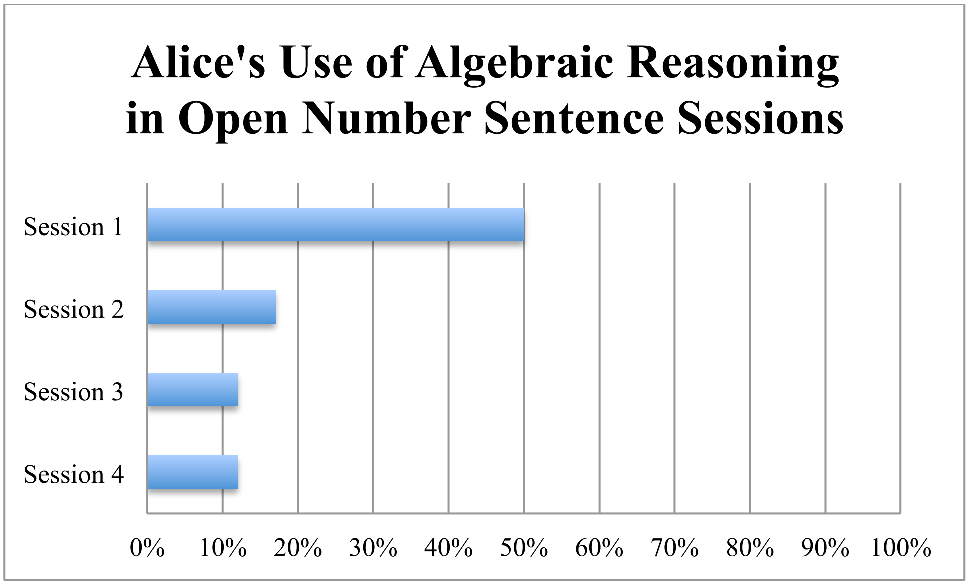


Figure 40. Alice's Use of Algebraic Reasoning During Open Number Sentence Sessions.

Alice used Algebraic Reasoning the most in Session 1. She used it by itself or with other CMIAS. Alice used only Algebraic Reasoning when she solved □ – 8 = -5 by solving it "backwards." The excerpt and drawing below (see Figure 41) illustrates how Alice changed the structure of the open number sentence and solved it "backwards."

A: (Writes -5 and then an 8 vertically on top. Then, she uses fingers to count and writes -13 in the box). I did negative thirteen in the box minus eight to give me negative five because I did the same thing as last time and did it backwards.

T: Ok. So, can you explain the backwards again?

A: I did negative five plus (points at equal sign) eight equals (points at minus sign) negative thirteen.

T: Ok. And, how did you get the negative thirteen?

A: Because, I did eight plus negative five.

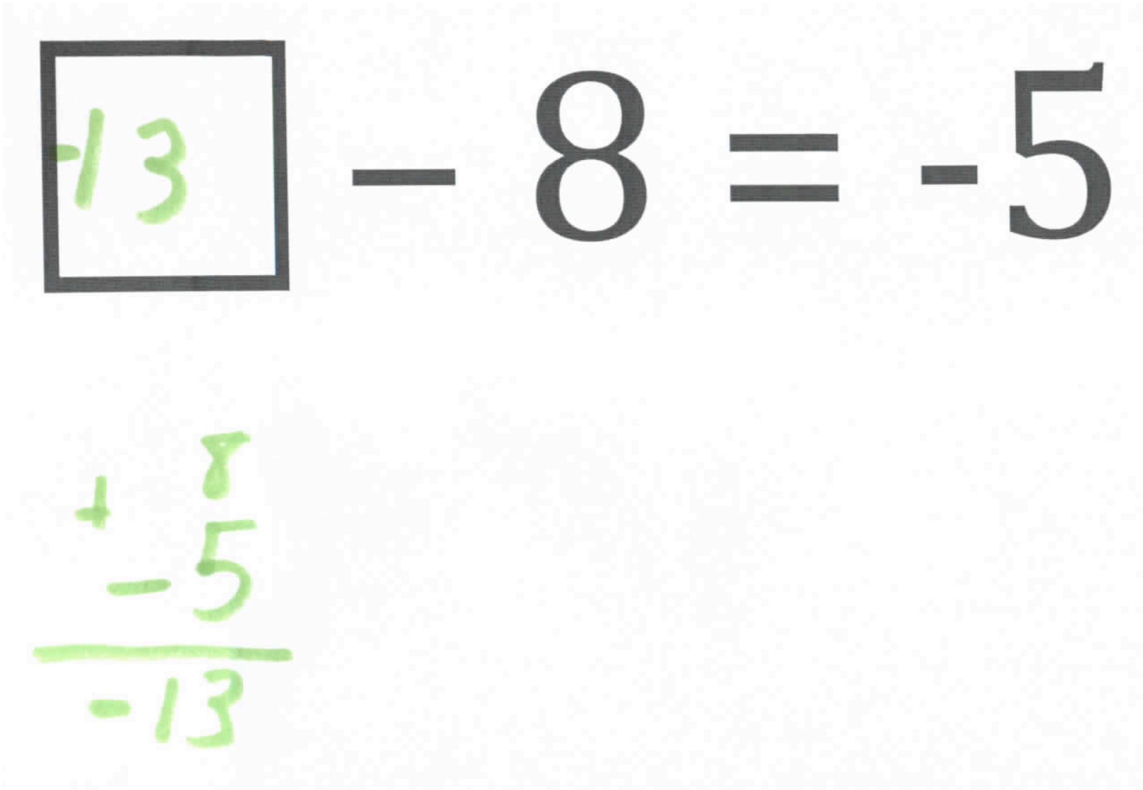


Figure 41. Alice's Drawing for Solving □ – 8 = -5.

Alice referenced in this excerpt that she solved it this away "again." Although she did not obtain the correct answer because she thought that 8 + -5 = -13, changing the structure of problem was a productive use of Algebraic Reasoning that she used several times in the session. For example, Alice also used Algebraic Reasoning paired with another other CMIASs, Translation in Session 1 to solve □ – -2 = 1 correctly (see Figure 42). Alice solved □ – -2 = 1 using Algebraic Reasoning and Translation. In the following excerpt, Alice used Algebraic Reasoning when she solved by changing the structure of □ – -2 = 1 to 1 + -2. She used Translations when she "counted up" from 1 to -1.

A: (Writes a vertical problem). I did one... I did it backwards. I did one plus negative two. And, I got negative one.

T: Ok. Can you explain that?

A: Negative one. I did one plus two, negative two I counted it up. I counted one plus negative two up and I got negative one.

T: What do you mean that you counted from one to negative two up? Can you explain that?

A: Well, I had negative two is below zero. So, I did negative two and I added one and I got one.

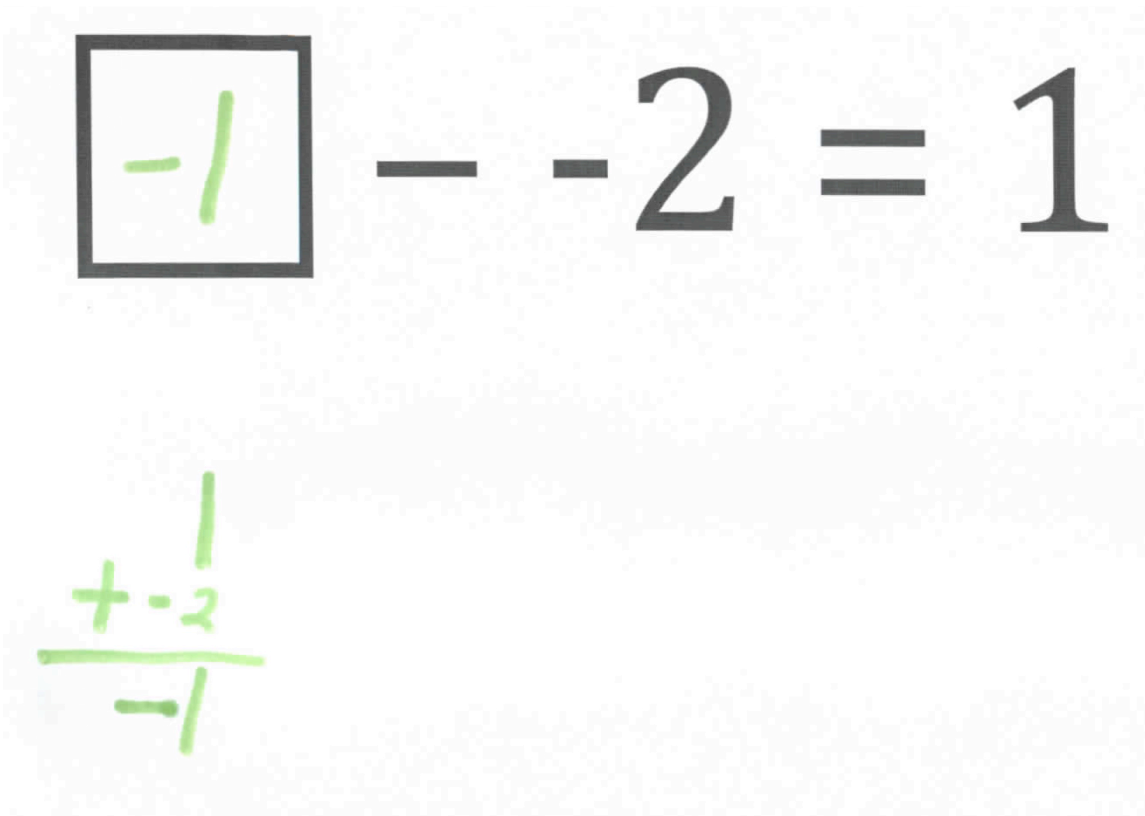


Figure 42. Alice's Drawing for Solving □ – -2 = 1.

Alice's use of Algebraic Reasoning by Session 4 was more subtle than it was in the Session 1. That is, Alice did not explicitly reference that she was solving the problems "backwards" when she changed the structure of the problem in Session 4. Also, in the last session, Alice used Algebraic Reasoning with other CMAISs only. For example, in

Session 4 Alice used Algebraic Reasoning and Proceduralization when she solved □ + -3 = 7:

> (Writes 10 in the box.) I did ten because it'd, since it's positive (points at 7) plus negative (points at -3), it'd be just like subtracting ten. Ten minus three is seven.

Alice used Algebraic Reasoning when she changed the structure of the problem to 7 + 3. For example, she stated, "I was doing seven (points at 7) plus three (points at -3) and it equals ten (points at 10)." Alice used Proceduralization when she stated a rule that adding -3 is "just like subtracting ten."

Summary of Alice's CMIAS Use

Although Alice posed different stories with different contexts and in different ways, Alice's use of Bookkeeping remained consistent and dominant from Session 1 to Session 4 (see Table 22). Overall, Alice's use of Algebraic Reasoning declined from Session 1 to Session 4. Alice's use of Analogy remained somewhat consistent from Session 1 to Session 4. However, Alice's use of Proceduralization increased from Session 1 to Session 4. Alice's use of both Bookkeeping and Counterbalance increased from Session 1 and 3, but declined in Session 4. And, Alice's use of Translation peaked in Session 2 and declined in use after that.

Alice used only Bookkeeping to pose stories during the Individual Context Sessions; yet, Alice used nearly all of the CMIAS in the Individual Number Sentence Sessions. Alice did not use Relativity in the Context Sessions nor the Open Number Sentence Sessions. Alice's most utilized CMIAS during the in the Open Number Sentences sessions changed throughout sessions. For example, she used Algebraic Reasoning the most in Session 1, Translation the most in Session 2, Counterbalance the most in Session 3, and Proceduralization in the most in Session 4. Although the CMIAS she used the most the in the open number sentence sessions changed, she remained consistent with CMIASs that she used in the context interviews. Overall, it appears that solving open number sentences gave Alice the opportunity to demonstrate flexibility with her thinking.

Table 22

Alice's Overall Use of the CMIAS Across Individual Sessions

	Individual Context Sessions				Individual Open Number Sentence Sessions			
	1	2	3	4	1	2	3	4
Bookkeeping	100%	100%	100%	100%	10%	26%	32%	12%
Counterbalance	0%	0%	0%	0%	15%	39%	48%	44%
Relativity	0%	0%	0%	0%	0%	0%	0%	0%
Translation	0%	0%	0%	0%	35%	61%	32%	28%
Proceduralization	0%	0%	0%	0%	30%	43%	40%	72%
Analogy	0%	0%	0%	0%	24%	25%	22%	25%
Algebraic Reasoning	0%	0%	0%	0%	50%	17%	12%	12%

CHAPTER V

JACE'S LEARNING OF THE CMIAS & THREE INTEGER ADDITION AND SUBTRACTION PROBLEM TYPES

This chapter begins with a brief discussion of Jace's CMIAS use and learning of CMIAS. This brief discussion is provided to give an overall background of Jace's thinking and learning. Then, this chapter narrows and provides a snapshop of Jace's learning by describing his changes in mathematical discourse for three different open number sentences.

Jace's Use & Learning of the CMIAS

Learning is defined as a change in mathematical discourse in this study (Sfard, 2008). And, this study interpreted the CMIAS use of the students as a way to describe the students' narratives, a central tenet of mathematical discourse. Thus, if we describe the narratives and changes in these narratives, an aspect of learning is discussed.

Jace's overall CMIAS use is shown in Table 23. As illustrated in the Table 23 below, Jace used Bookkeeping in the Individual Context Sessions, but did not use Bookkeeping during the Individual Open Number Sentence Sessions. Although Jace used Algebraic Reasoning, Counterbalance, and Translation in the Individual Sessions, these CMIAS were less utilized CMIAS when compared to his use of Proceduralization and Analogy.

For example, Jace used Analogy 68% of the time in Individual Open Number Sentence Session 4, while Jace used Counterbalance only 12% of the time in this session.

Table 23

Jace's Overall Use of the CMIAS Across Individual Sessions

	Individual Context Sessions				Individual Open Number Sentence Sessions			
	1	2	3	4	1	2	3	4
Bookkeeping	29%	71%	100%	88%	0%	0%	0%	0%
Counterbalance	0%	0%	0%	0%	0%	17%	24%	12%
Relativity	29%	14%	0%	0%	0%	0%	0%	0%
Translation	43%	14%	0%	13%	30%	22%	20%	28%
Proceduralization	0%	0%	0%	0%	65%	70%	48%	60%
Analogy	0%	0%	0%	0%	50%	48%	52%	68%
Algebraic Reasoning	0%	0%	0%	0%	30%	26%	40%	32%

The next section will paint a small picture of Jace's use of the CMIAS by describing the changes in the uses for two of his most utilized CMIAS (i.e., Proceduralization and Analogy) and one of his less utilized CMIAS (i.e., Counterbalance).

Jace's Use & Learning of Proceduralization

Session 1

When Jace used Proceduralization in Session 1, his procedures often focused on the role of the plus and minus symbols. For example, in the following excerpt Jace stated that you "don't need the addition symbol":

> (Draws boxes.) mmm, never mind. (Crosses the boxes off.) Well, if there was ... Because it's a negative number it's pretty much saying ...well, you don't need the addition symbol. So it would be sixteen minus twelve, which (draws 16 – 12 = 4) sixteen minus twelve would be four.

In Figure 43, Jace demonstrated his ignoring of the plus sign by crossing off the "+" and circling the negative sign of -16.

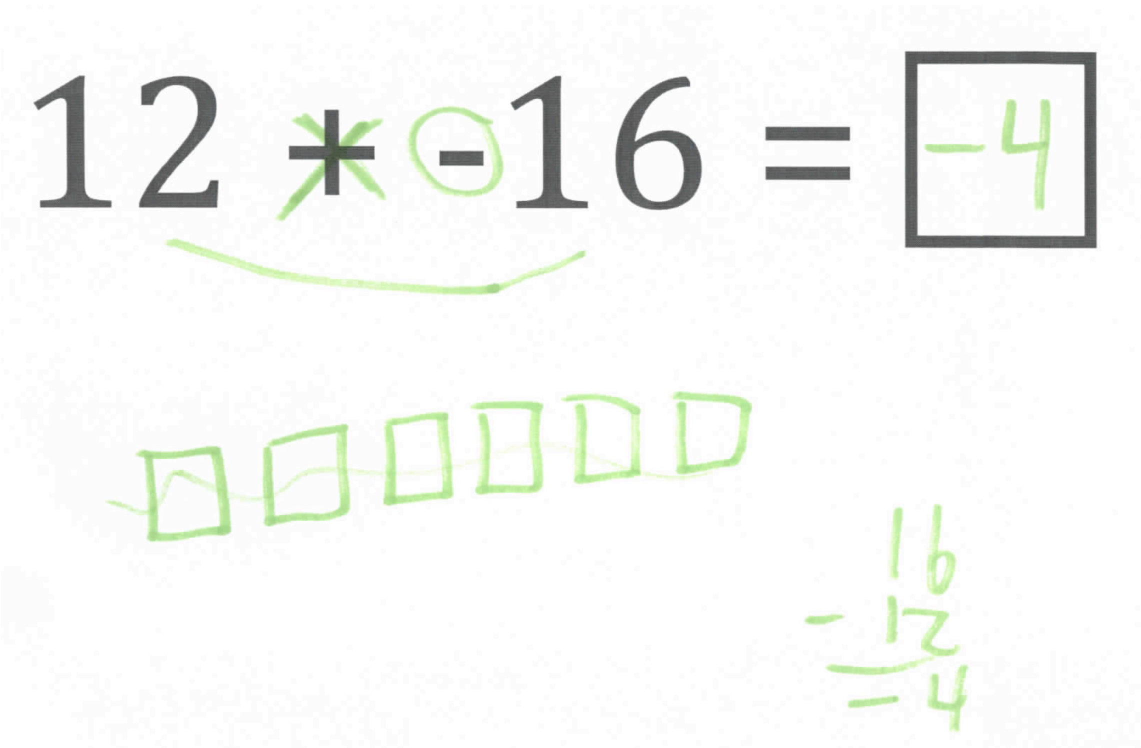


Figure 43. Jace's Drawing for Solving 12 + -16 = □.

In this session, Jace utilized Proceduralization with other models, like Algebraic Reasoning. For example, when solving -7 + □ = -2, Jace stated, "Well, this one is basic, I guess. Like basic stuff in your head, like Kindergarten. You could do seven minus two equals five or, but in this case it would be negative five." He utilized Algebraic Reasoning here when he changed the structure of the open number sentence. At first he thought the answer was -5 and then he changed his answer as he used Proceduralization when he stated, "But, I realized that, like earlier, I said if you have negative plus a negative you don't need that plus symbol. But, in this case you do. So that would have to be real five, and not like negative five."

Session 2

Jace used Proceduralization the most in Session 2, 70% of the time, which is an increase in use from the previous session. He utilized Proceduralization by itself and with other CMIAS. He used Proceduralization both correctly and incorrectly. He also extended the rules he developed from the first session. One of the rules that Jace used in the first session was that the "addition symbol" could be ignored. In this session, He extended this reasoning to include that the "minus symbol" could also be ignored (see Figure 44). For example, when solving □ – -3 = 2, Jace stated, "Like what I said earlier in a different problem and just use this one (points at the negative symbol). So five minus three equals two."

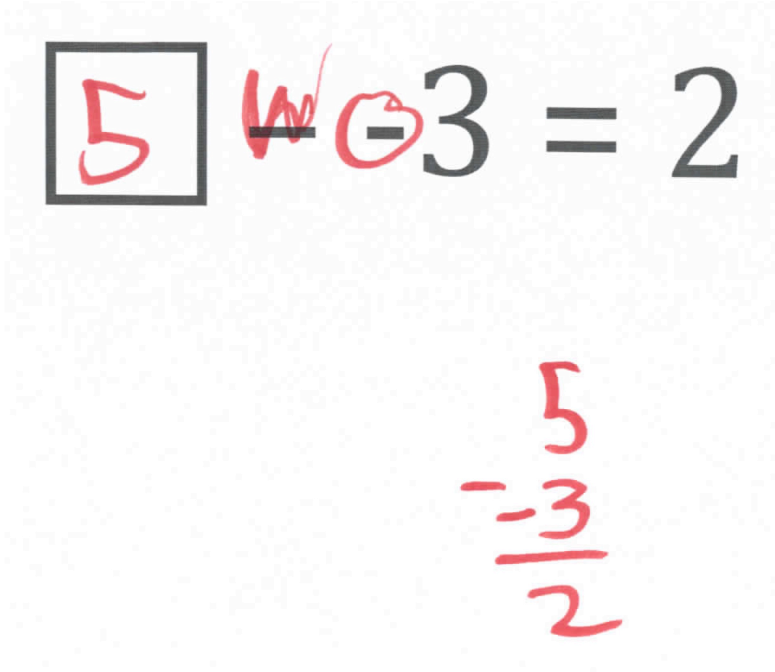


Figure 44. Jace's Drawing for Solving □ – -3 = 2.

Although Jace used Proceduralization incorrectly, like ignoring minus symbols, Jace also used Proceduralization correctly. For example, Jace used Proceduralization to solve 20 + -33 = □:

> It says twenty plus negative thirty-three. But, if you are doing a subtraction problem, then the bigger number has to be in front. Thirty-three minus twenty equals thirteen…which [you have to] add a negative symbol, change that to addition and add a negative symbol. If there is one negative…in the problem and the number is bigger than the [positive number], then you know you are still going to have an answer that is negative number.

In this excerpt, Jace first described a rule he used to solve the number sentence. He then concluded with a generalized rule.

Session 3

Proceduralization was the second most utilized CMIAS by Jace this session, with the most utilized CMIAS being Analogy. His use of Proceduralization decreased, using Proceduralization 48% of the time. Jace used Proceduralization by itself and alongside other CMIAS. Jace used only Proceduralization when he solved -4 – 10 = □:

J: (Draws vertical problem.) Ten minus four equals six (writes 10 – 4 = 6). So if you did ... if you flipped these two around (draws double arrows connect 10 and 4) and you made this negative (points at 4), it would be negative four minus ten equals negative six. Yeah. So, negative six goes in the box.

T: Negative six. Can you explain how you thought that through again?

J: Because ten minus four equals six. And, four minus ten would equal negative six. And, this is the number (points at -4) that's less value and it's the negative number, so it's basically, four minus ten, which equals six.

Again, open number sentences that were the problem types -a – b were challenging for Jace. Here he drew upon a Proceduralization that -4 – 10 could be solved by 10 – 4. His explanation was focused on the procedure that he applied when solving this.

Jace used Proceduralization with other CMIAS. For example, Jace used Analogy and Proceduralization when he solved □ – -5 = 0. Jace began with Analogy when he compared 5 – 5 to -5 – -5 and he stated, “Five minus five (writes 5 – 5) equals zero. And, you can do that if you want because they are both the same number and it doesn't matter if you put it in there or not, it would still be five minus five. And, you would still get zero if you added a negative symbol.” Jace then used Proceduralization when he stated a rule about symbols, “Because ... Five goes right there because there's a negative symbol and a subtraction symbol. And, that's a five right there (points at -5). So, all you are doing is five minus five equals zero."

Session 4

Proceduralization was the second most utilized CMIAS this session, with Analogy the most utilized this session. Jace's use of Proceduralization increased from Session 3, using Proceduralization 60% of the time. Jace used Proceduralization when he solved 1 – □ = 3. As he began solving this open number sentence he reflected, "Remember when we did this one time and then I couldn't figure it out because it was a subtraction and then ...but, now I do because of the Group Sessions and all that." Then, Jace drew a horizontal problem 1 – -2 = 3 and stated, "One minus negative two equals three." When asked how he knew the answer he reasoned:

> In a problem like this when it's one minus something or like a low number minus something is a bigger number than you first had (points at 1) it's kind of confusing at first. That's why I couldn't be able, I wasn't able to this problem before. But now I realize that if you have a regular number and you're trying to take a away a negative number, you don't even have a negative number. So that's basically just adding.

In this excerpt, Jace illustrated Proceduralization because his explanation was focused on a procedure. Although Jace used Proceduralization to solve 1 – □ = 3, Jace mostly used Proceduralization with other CMIAS in this session. For example, Jace used Proceduralization and Analogy when he solved 2 – -3 = □:

> (Drew horizontal problem. Wrote 2 – -3 = 5). It's five because two you don't have a negative number, so if you are taking away a negative number you are adding.

Like the last couple of problems. So, two plus three equals five. I think about it like taking the negative symbol and turning it and making that an addition symbol. So ...(writes five in the box).

In this problem, Jace again applied Proceduralization because he used a rule. However, Jace also used Analogy because he compared this problem to previous problems.

Conjecture about the Influence of the Group Sessions on Jace's Learning

The Group Sessions[3] were designed to promote thinking about other CMIAS than Proceduralization. However, Proceduralization remained the most ulitized CMIAS by Jace. In fact, all three of the students, as they participated in the Group Sessions, would often digress to talk about "procedures" and "rules." Often the teacher-researcher and witness saw that the "rules" that the students developed interfered with the discussion in Group Sessions and learning to reason with the CMIAS. Because of this, the design of the following Group Session was often influenced by the students' use of Proceduralization. For example, a context would be selected to help the students think about their "rules" differently. For example, in Group Session 6, the context of temperature rising and falling was selected to help the students think differently about the rules they had developed for the problem type $-a - b = \square$. Proceduralization was not only the preferred CMIAS of Jace, that would trump the other CMIAS, but Proceduralization was the CMIAS that all of the students seemed to want to progress to. Thus, the students' preference for Proceduralization seemed to drive the development of the Group Sessions and the learning of Procedurlizations would have happened without the Group Sessions. However, the relationship between the Group Sessions and learning of Proceduralization

[3] Analyses of the Group Sessions were beyond the scope of this dissertation study.

was cyclic, and the Group Session did influence the learning of Proceduralization. Not only was the development of the Group Session dependent sometimes on the students' use of Proceduralization, but in terms of learning, the Group Sessions' influence on Jace's learning of Proceduralization seemed to provide opportunity of time for learning. Jace, as well as the other students, wanted to use Proceduralization, and were afforded time to think differently about integer operations and develop Proceduralization with productive uses.

Jace's Use & Learning of Analogy

Session 1

Analogy was the second most utilized CMIAS that Jace in Session 1, using it 50% of the time. Utilizing Analogies in this session were both productive and limiting. For example, Analogy limited Jace in this session in that he was not able to solve the open number sentence 1 – □ = 3. He stated, "It's kind of confusing because it would, to me at least... If you have one minus something it would be zero or in the negative numbers." He then compared 1 – □ = 3 to 3 – 2 when he stated, "if you flip this around, three minus two equals one." However, utilizing Analogy was productive for Jace, as well, in this session. For example, Jace drew upon Analogy when he solved □ – 8 = -5 (see Figure 45). Jace used Analogy when he stated, "If you have eight minus three it would be five. So, basically you are just flipping these around…I know eight minus three would equal five…but the answer is negative five." Here Jace is directly comparing 8 – 3 = 5 to 3 – 8 = -5.

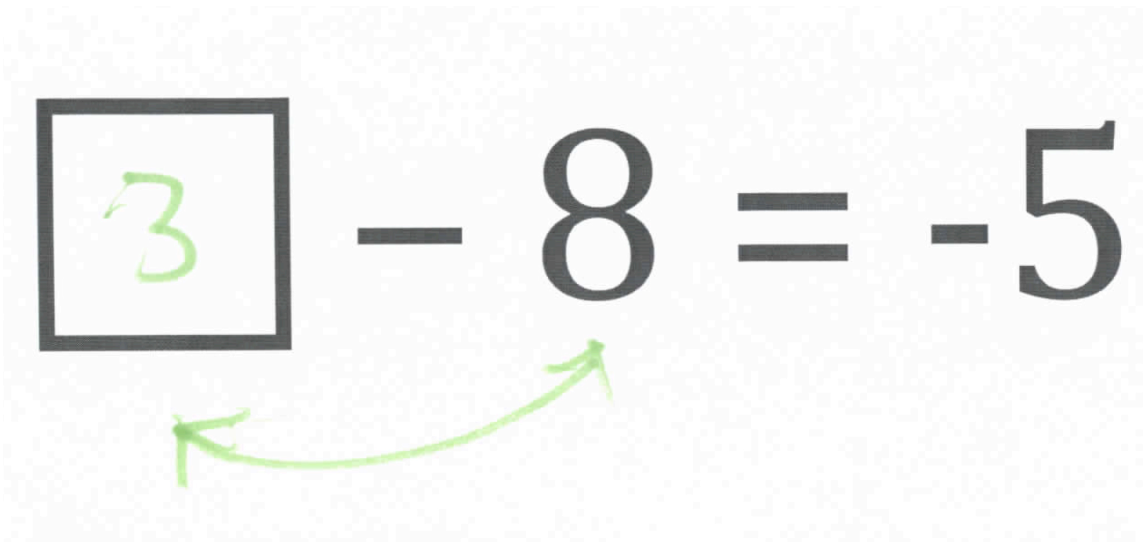


Figure 45. Jace's Drawing for Solving □ – 8 = -5.

Session 2

Analogy was the second most utilized CMIAS by Jace in this session, using it 48% of the time. His use of Analogy in Session 2 remained somewhat consistent to his use in Session 1.

Jace used Analogy paired with other CMIAS, with one exception. He used only Analogy when solving □ + -9 = -21:

> I put negative twelve because, or, yeah, negative twelve. Because twelve plus nine equals twenty-one. And, the nine is a negative number (points at -9), so the, and and the twenty-one is a negative number (points at -21). So you have to start with a negative number too.

In this excerpt, Jace compared 12 + 9 to -12 + -9, which was an Analogy. However, Jace mostly use Analogy paired with other CMIAS. For example, Jace used Analogy and Proceduralization together he solved □ – -4 = 0.

> (Draws a vertical problem.) So it says the answer is zero, and this number is four (points at -4). That means that, well this number is negative four (points at negative four). So that means that the first number has to be four too because four minus four equals zero. And, we don't need that (uses marker to cross of minus sign) and we just need this (circles the negative symbol).

He used Analogy when he compared 4 – 4 to -4 – -4 and he used Analogy when he was stating rules about the minus symbols (see Figure 46).

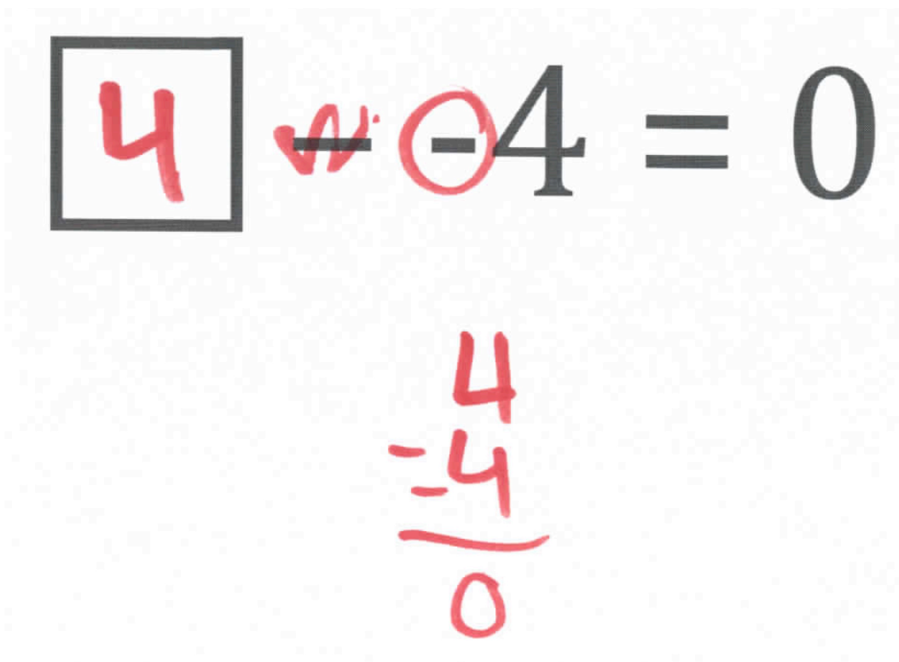


Figure 46. Jace’s Drawing for Solving □ – -4 = 0.

Session 3

Analogy was Jace’s most utilized CMIAS in this session, using it 52% of the time in Session 3. Jace’s use of Analogy in Session 3, remained somewhat consistent with his use in Session 1 and Session 2.

Jace used Analogy consistently in this session with and without other CMIAS. Although Jace was not able to solve -10 – □ = 11 in this session, he used Analogy as he attempted to make sense of this open number sentence and solve it.

> I'm not really sure about this one either. Because if you negative ten minus negative one, then that would get you negative nine. I'm not sure how to ... It's kind of hard to describe because if you subtract a whole number, like let's say you did negative ten minus four, that would get you negative six because four is a whole number.

In this excerpt Jace compared -10 – □ = 11 to -10 – -1 = -9 and -10 – 4 = -6. Jace typically struggled with the problem type -a – b and he used Analogy to other integer number sentences to make sense of this this challenge. Again, Jace used Analogy to whole number sentences frequently. Sometimes he used Analogy productively and sometimes he used it unproductively. Analogy was utilized unproductively when Jaced solved □ – -3 = 1:

> (Draws vertical problem. Writes 4 – 3 = 1.) If you do, four minus three then you'll get one. And if you do four minus three that would still be one. So, I think the answer here (points at box) is four.

Here Jace used Analogy to compare 4 – -3 to 4 – 3, which was not productive. However, Jace used Analogy productively in this session. Analogy was used productively when Jace solved -12 – -4 = □:

> (Draws vertical problem. Writes 12 – 4 = 8. Then, adds negative symbols to 12, 4, and 8.) I think it's negative eight because you're subtracting like you would with regular numbers. So that would be twelve minus four equals eight, but since they

are both negative numbers it would be negative twelve minus negative four equals negative eight.

In this excerpt Jace made use of Analogy productively when he compared -12 – -4 to 12 – 4.

Session 4

Although Analogy was the most utilized CMIAS this session again; however, his use of Analogy increased from Session 3 from use 52% of the time to 68%. Again, Jace used Analogy by itself or with other CMIAS. Jace used only Analogy, with analogies to whole numbers, when he solved □ – -3 = 0, “I think it's negative three because if you did three minus three equals zero, so negative three minus negative three equals zero.” Jace also used only Analogy when he solved -2 + □ = -10, “I think the answer for box is negative eight. Because two plus eight equals ten. So, negative two plus negative eight equals ten.”

Although Jace used only Analogy in the previous examples, he often used Analogy paired with other CMIAS in this session, and Integer Analogies. The previous example demonstrated how Jace made Whole Number Analogies; however, Jace also made Integer Analogies or compared a number sentence with negative integers to a different number sentence with negative integers. For example, when Jace solved -5 – 4 = □ he used Proceduralization, Algebraic Reasoning, and Analogy, with an analogy to an integer addition and subtraction problem. This problem type was notoriously challenging for Jace and other students. He began solving this problem by drawing upon Proceduralization, “Like I'm thinking maybe ... I'll write it down, but I think that negative five minus four (start drawing horizontal problem) would equal negative one. Because, just like last problem, it's a negative number and a subtraction. So, but, I don't know."

Then, he used Algebraic Reasoning next by changing the structure of the problem:

> Because... ah maybe not. You know what... I think it's nine because ...if you have a negative five and you flip the problem around. So four minus negative five that would be five because you are taking away a negative number from the four even though you don't have a negative number. So, it would be plus instead of minus a negative.

He then turned to Analogy, by comparing -5 – 4 = □ to 4 – -5 = = □:

> Because it's ... to be negative one it would have to be negative five minus negative four, because five minus four equals one and then they're all negative numbers. But, since it's not then it's going to be a different answer.

Because Jace knew the answer to 4 – -5 was 9, he then concluded that -5 – 4 equaled 9. Although this was still the mathematically incorrect answer, based on how he typically reasoned about these problem types he would have typically responded that -5 – 4 would have equaled -1.

Conjecture about the Influence of the Group Sessions on Jace's Learning

Analogy was also a CMIAS not intentionally promoted in the Group Sessions. Yet, Analogy remained a prominent CMIAS for Jace, similar to Proceduralization, throughout the Individual Sessions. Unlike Procedurlization, the Group Sessions were not intentionally designed to support discussion about Analogies. Because Proceduralization seemed to be the CMIAS that Jace progressed towards throughout the Individual Sessions, and similarly for the other students, it seems natural that Analogy would also be a prominently utilized CMIAS. Jace, and the other students, have been drawing upon whole numbers and operations with them for their lifetime and whole number instruction has the focal point of the school mathematics instruction. As the students were challenged to think differently about number in the Group Sessions, they would often turn to Proceduralization, and pair it with an Analogy to inform those developing "rules." The Group Sessions did not appear to influence the development of Analogy, other than the opportunity of time. It seemed that Jace, and the others, wanted to build off of their strongly rooted whole number knowledge. They seemed to prefer Analogy in the Individual Sessions to the recently learned opportunities of contexts and other CMIAS provided in the Group Sessions.

Jace's Use & Learning of Counterbalance

Session 2

Jace did not use Counterbalance in Session 1 and using Counterbalance only 17% of the time in Session 2. In Session 2, Counterbalance was the least utilized CMIAS by Jace, sans Bookkeeping and Relativity, which he did not use at all. For example, he used Counterbalance when he solved $5 + \square = -3$. In the follow excerpt of transcript Jace discussed the magnitudes of 5 and -8 to solve $5 + \square = -3$:

J: (Writes a vertical problem.) I did... I was kind of confused at first because it was a five (points at 5) and then it changed to a negative number (points at -3). And then, I figured out that eight is greater than five, so that'd be negative eight plus five is in the ... is negative three because eight is bigger than five. And, eight is the negative number.

T: So you think what goes inside here (points at box)?

J: Eight. Negative eight. (Writes -8 in the box.)

Because Jace's reasoning was magnitude-based, comparing the two different quantities of 5 and -8, it was considered that he used Counterbalance. This was the first time that Jace demonstrated this type of reasoning in an individual session.

Session 3

Compared to Session 2, Jace increased his use of Counterbalance in Session 3. His Counterbalance use increased from 17% to 24%.

The following excerpt of transcript highlights how Jace used Counterbalance as he solved -5 – 3 = □:

J: (Draws a vertical problem. -5 – 3 = -2) I think the answer would be negative two because the negative number is bigger than the one that's not a negative number, so your answer would still be negative.

T: Ok. Can you explain how you were thinking about that?

J: Like ... This number is the negative number (points at -5 and uses marker to circle it) and this number is the regular number (circles 3 with marker). Since this number (points at -5) is the negative number and this one's not (points at 3), it's still going to be in the negative number because this one is bigger (points at -5).

T: Ok. So you think the answer's ...

J: Negative two.

Jace compared the magnitudes of -5 and 3 to solve -5 – 3 = □. Although he did not obtain the correct answer, he used Counterbalance to make sense of this open number sentence. Again, this type of open number sentence was typically challenging for Jace. This was one of the two number sentences where Jace used only Counterbalance. Jace also used Counterbalance when he solved -18 + 12 = □. His reasoning was similar as above for solving -18 + 12 = □, “I ... at first I did eighteen minus twelve equals six. But, then I added ... then I made it so it was negative eighteen plus twelve and then my answer is negative six because twelve is less than eighteen and eighteen's the negative number so you're still in the negatives.” This way of reasoning was productive for -18 + 12 = □, but not for -5 – 3 = □.

For the rest of Jace’s Counterbalance use this session, he used it paired with other CMIAS. For example, Jace used both Proceduralization and Counterbalance when he solved □ + 25 = -2. Jace began with Proceduralization when he stated what he did and focused on the procedure, “(Writes vertical problem. Writes 27 minus 25 equals 2 vertically. Then, adds negative symbols to 27 and 2 to make the problem -27 - 25 = -2.) This needs to be negative. And, that's a minus sign. So I think the answer is twenty-seven for box (draws arrow from -27 to box).” Jace continued his explanation with Counterbalance when he reasoned, “because twenty-seven minus twenty-five equals two. But, it's negative twenty-seven because twenty-five is a regular number and twenty-seven is greater than twenty-five.”

Session 4

Jace used Counterbalance the least of the utilized CMIAS in this session, using Counterbalance 12% of the time. His use of Counterbalance decreased from Session 3 from 24% to 12%.

Jace used Counterbalance when he solved 12 + -16 = □. Jace’s reasoning for solving 12 + -16 = □ was magnitude-based as he compared the two quantities:

Sixteen is bigger than twelve. And, sixteen's the negative number. So sixteen minus twelve equals four. And, it's got to be negative four because the negative number's bigger than the regular number.

Jace also used Analogy and Counterbalance when he solved -20 + 15 = □:

Because twenty minus fifteen equals five. But, it's negative twenty plus fifteen and the negative twenty is bigger than ... well, twenty is bigger than fifteen. And, twenty's the negative number. So you are still going to be in the negative numbers.

He first used Analogy to reason that -20 + 15 = -5 because 20 – 15 = 5. He then used Counterbalance when he compared the magnitudes of -20 and 15, and reasoned, "twenty is bigger than fifteen."

Conjecture about the Influence of the Group Sessions on Jace's Learning

In between Individual Sessions 1 and 2 were the first three Group Sessions. Of these Group sessions, the Group Session 3 used contexts to introduce Counterbalance. Notably, Jace's use of Counterbalance increased after that. Similarly, in between Individual Sessions 2 and 3, were three Group Sessions. Of these Group Sessions, Group Session 4 and 5 focused on promoting Counterbalance through the context of a game. Also, notably, Jace's use of Counterbalance was the highest after this set of Group Sessions. In between Individual Sessions 3 and 4 there were three Group Sessions that used other contexts to promote other CMIAS, like Translation and Relativity. After these Group Sessions, Jace's use of Counterbalance decreased. Perhaps this is due to Counterbalance not being re-enforced in these Group Sessions or the others being more dominant since they had been more recently utilized.

Jace's Correctness

Jace's correctness improved over the 12-week period for the Individual Open Number Sentence Sessions. The green font in Figure 35 represents problems that were eventually solved correctly in the Individual Open Number Sentence Sessions, with all solutions recorded even the incorrect solution. Although Jace's correctness improved, it is notable to observe how long it took for Jace to make sense of the subtraction problems despite the support of the teaching experiment, with conceptually-based Group Sessions. Across the Individual Sessions 1, 2, 3, and 4, Jace became better at solving open number sentences (see Figure 47). He increased from getting 50% of the open number sentences correct in Session 1 to 98% of the open number sentences correct in Session 4. In Figure 35, the open number sentences are matched up by problem type across the four sessions. Jace's correct answers are in green and his incorrect answers are in red. All of the answers that Jace provided during the session are listed. For example, in Individual Session 4 for -15 – -4 = □, Jace first stated -19, which was incorrect. He then changed his answer to -11, which was correct. Both of these solutions, -19 and -11, are listed in the cell, but because Jace's final answer was correct, it considered that he answered that open number sentence correctly.

Session 1		Session 2		Session 3		Session 4	
-20 + 15 = □	-5	-16 + 4 = □	-12	-18 + 12 = □	-6	-20 + 15 = □	-5
12 + -16 = □	-4	20 + -33 = □	-13	15 + -24 = □	-9	12 + -16 = □	-4
-4 + □ = 10	14	-6 + □ = 15	21	-3 + □ = 14	17	-4 + □ = 10	14
-7 + □ = -2	-5, 5	-6 + □ = -1	5	-9 + □ = -3	-6, 6	-7 + □ = -2	5
□ + -3 = 7	10	□ + -2 = 17	19	□ + -4 = 13	17	□ + -3 = 7	10
□ + 13 = -5	-18	□ + 19 = -4	-23	□ + 25 = -2	-27	□ + 13 = -5	-18
-8 + -7 = □	-1	-12 + -5 = □	-17	-17 + -6 = □	-23	-8 + -7 = □	-15
-2 + □ = -10	-8	-4 + □ = -19	-15	-5 + □ = -21	-16	-2 + □ = -10	-8
□ + -9 = -16	-7	□ + -9 = -21	-12	□ + -9 = -17	-8	□ + -9 = -16	-7
10 – 12 = □	-2	5 – 9 = □	4, -4	12 – 18 = □	-6	10 – 12 = □	-2
1 – □ = 3	not possible	4 – □ = 6	not sure	3 – □ = 4	not sure	1 – □ = 3	-2
-5 – 4 = □	-1	-9 – 8 = □	-1	-5 – 3 = □	-2	-5 – 4 = □	-1, 9
2 – -3 = □	-1	3 – -4 = □	-1	1 – -3 = □	-2	2 – -3 = □	5
-1 – □ = 8	7	-2 – □ = 9	11	-2 – □ = 10	12	-1 – □ = 8	9, not sure
2 – □ = -10	8	6 – □ = -10	-16, 16	4 – □ = -12	16	2 – □ = -10	12
□ – -1 = 6	-7, not sure	□ – -1 = 4	5	□ – -2 = 5	7	□ – -1 = 6	5
□ – 8 = -5	3	□ – 9 = -3	6	□ – 6 = -2	4	□ – 8 = -5	3
-15 – -4 = □	-11	-11 – -2 = □	-13	-12 – -4 = □	-8	-15 – -4 = □	-19, -11
-12 – □ = -13	-1	-15 – □ = -16	-1	-10 – □ = -11	not sure	-12 – □ = -13	1
□ – -2 = 1	3	□ – -3 = 2	5	□ – -3 = 1	4	□ – -2 = 1	-1
		□ – -4 = 0	4	□ – -5 = 0	5	□ – -3 = 0	-3
		12 + □ = 8	-4	15 + □ = 9	-6	17 + □ = 8	-9
		5 + □ = -3	-8	8 + □ = -5	13, -13	6 + □ = -2	-8
				□ + 2 = 0	-2	□ + 4 = 0	-4
				-4 – 10 = □	-6	-2 – 8 = □	-10
Percent Correct	55%		57%		64%		92%

Figure 47. Jace's Answers to Open Number Sentences.

A close examination of Figure 47, illustrates that for the most part, Jace was generally able to do addition problems with integers prior to study and solve them consistently correct throughout the study. Yet, some problem types, and specifically subtraction problems, remained difficult for Jace throughout the Individual Sessions (see, e.g., -5 – 4 = □). Yet, other problem types, and specifically addition problems, Jace solved successfully across the four session (see., e.g., -4 + □ = 10).

While the previous portion of this chapter highlighted Jace's CMIAS use, this next part will focus on three problem types and highlight all four components of Jace's mathematical discourse (i.e., word use, visual mediators, narratives, routines). Jace's learning for three problem types across the four Individual Open Number Sentence Sessions (i.e., Sessions 1, 2, 3, 4) are highlighted next. The open number sentence types selected demonstrate a problem type that Jace solved correctly across the four sessions, a problem type that Jace solved incorrectly across the four sessions, and a problem type that Jace solved incorrectly but eventually solved correctly in the sessions. These problem types, respectively, are: -a + □ = b (a, b > 0 and b > a), -a – b = □ (a, b > 0 and a > b), and -a – □ = -b (a, b > 0 and b > a).

Describing the Learning of -a + □ = b (a, b > 0 and b > a)

Jace solved the problem type -a + □ = b (a, b > 0 and b > a) correct across all four sessions. Although he answered this problem type correctly across the sessions (see Figure 47), how Jace solved this varied across the sessions (see Figures 48, 49, 50, and 51). Figure 4 illustrates Jace's learning of problem type -a + □ = b (a, b > 0 and b > a) by including Jace's transcripts (word use), drawings (visual mediators), reasoning for solving the open number sentence (narratives), and describing how much Jace used that narrative and type of visual mediator in that particular individual session (routines).

Word Use

Jace's word use in Session 1 began with discussing how to draw a number line to model this number sentence. Jace began with verbally expressing, "I'm going to do the number line thing again" (see Figure 48). He then described the actions of his drawings. Jace's word use in Session 2 began with solving 16 + 5. As he continued his verbal explanation, he transitioned into talking about how to use the number line in the latter part of the explanation. This differs from Session 1, where his word use was initiated with number line discussion rather than serving as a justification. Then, in Session 3, none of Jace's word use included moving about a number line or distances on a number line. Instead, Jace's word use included the "commutative property." In Session 4, Jace was efficient in his word use for explaining how to solve the open number sentence. Also, his word use in this session centered on his generalizations for solving this type of problem. Across the four sessions, Jace called positive integers either "whole numbers" or "regular numbers."

Individual Session 1 -4 + □ = 10	Individual Session 2 -6 + □ = 15	Individual Session 3 -3 + □ = 14	Individual Session 4 -4 + □ = 10
J: I'm going to do the number line thing again. (Draws line with two tic marks at each end). I will just put negative ten here because that's all we really need. And, I will put ten right here. And zero right here. So to get from negative four to ten, you would have ...Well, you could do this first. You could put a four right here. (Draws a four above zero. Then draws a connecting line from four to ten). And, from regular four to ten would be six. And if you added four, from zero to ten. That would be just four there (draws four above the connecting line from 0 to 4). Four plus six equals ten. If you added another four, which is right here, (draws a connecting line from 0 to -4), Then that would be fourteen. So, negative four plus fourteen equals ten.	J: (Writes horizontally 15 + 6 = 21). I did fifteen plus six because the answer ... Since fifteen is a whole number and then that would be just regular fifteen, but you have to add six more because the six goes ... Hold on. Here I will draw you one. (Draws a number line.) So that would be negative six right here (draws the negatives to the right) and fifteen right here (draws the positives to the left with 15 and -6 each an equal distance from 0 in the drawing.) It would be fifteen (draws an arch from 15 to 0 and writes 15 above the arch). Plus (draws a "+" above the zero) another six (draws an arch from 0 to 6 with 6 above the arch). T: Ok. So what's the answer that goes in the box? J: Ah negative ... Wait. Twenty-one (Writes 21 in the box.) Just regular twenty-one.	J: (Draws a vertical problem first. Vertically writes 14 + 3 = 17. Then, draws an arrow to the box.) T: Ok. Can you tell me what the answer in the box is? J: Seventeen. T: Ok. Can you tell me what you were thinking? How you figured that out? J: Because negative three is basically box (points at box) minus three. So, I did fourteen plus three and I got seventeen. And, seventeen minus three (points at -3) equals fourteen (points at fourteen). It's kind of like the commutative property. T: Oh. Ok. Can you explain the commutative property? J: You just flip it around and you still get the same answer. Like fourteen minus three, I mean fourteen plus three is seventeen. And, seventeen minus three is fourteen.	J: (Draws a horizontal problem. And writes 14 in the box.) T: Ok. How'd you get 14? J: Fourteen minus four equals ten. Fourteen plus negative four or negative four plus fourteen will equal ten. Because when you take a negative number and add it to a regular number, you are just subtracting. Instead of negative four plus fourteen, you can do fourteen minus four.
Word use begins with discussion on number line and is focused on this through the entirety of the transcript.	Word use begins an addition problem and then transitions to inclusion of the number line in the latter half of the explanation.	Word use is centered on the structure of the problem and includes reference to commutative property.	Word use is centered on the strategy or procedure utilized.

Figure 48. Jace's Word Use for Solving -a + □ = b (a, b > 0 and b > a).

Visual Mediators

Jace's visual mediators changed across the four sessions (see Figure 49). In Session 1, Jace drew an empty number line with three distances highlighted. In Session 2, Jace again drew a number line. However, this time Jace only used two the distances on both sides of the zero on the number line, rather than multiple distances. This change may point to Jace becoming familiar with using distances between or to zero to make sense of integer addition and subtraction. In Session 3, Jace did not draw a number line. Instead, Jace drew only a vertical number sentence. In Session 4, Jace drew only a horizontal number sentence. This my point to Jace no longer needing to draw upon the number line and becoming more efficient.

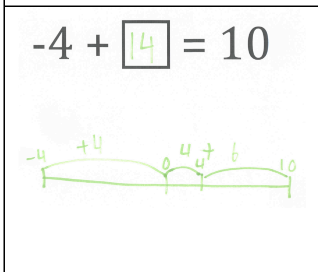
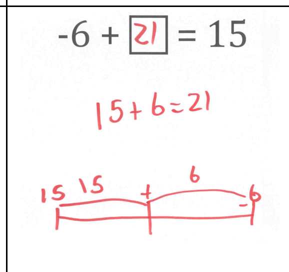
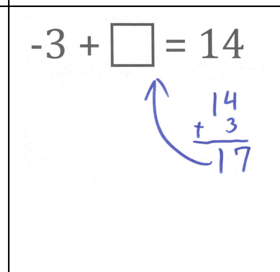
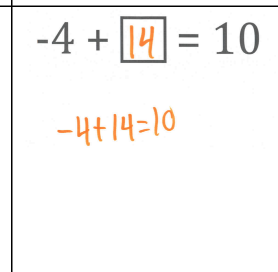
Individual Session 1 -4 + □ = 10	Individual Session 2 -6 + □ = 15	Individual Session 3 -3 + □ = 14	Individual Session 4 -4 + □ = 10
			
Empty number line partitioned into three distances, negatives on the left and positives on the right.	Empty number line partitioned into two distances, negatives on the right and positives on the left.	Vertical number sentences, with no negative integers.	Horizontal number sentence, with negative integers.

Figure 49. Jace's Visual Mediators for Solving -a + □ = b (a, b > 0 and b > a).

Narratives

Jace's narratives changed across the sessions as well (see Figure 50). In Session 1, it was considered that Jace used translation between numbers and the distances on a number line to solve the open number sentence. In Session 2, it was considered that Jace again used distance on a number line. However, in Session 2 Jace seemed to be also drawing upon some algebraic reasoning by changing the structure of the number sentence. By Session 3, Jace no longer used movements between numbers, but only used algebraic reasoning, or a structure change, to solve this open number sentence. In Session 4, Jace again used algebraic reasoning, but also used a rule that he had developed and constructed an analogy to whole numbers. Given that Jace used movements and distances on a number line in the first session and a rule he developed paired with an analogy and algebraic reasoning in the last session, this may point to Jace becoming more flexible with his reasoning.

Individual Session 1 -4 + □ = 10	Individual Session 2 -6 + □ = 15	Individual Session 3 -3 + □ = 14	Individual Session 4 -4 + □ = 10
Translation	Algebraic Reasoning Translation	Algebraic Reasoning	Algebraic Reasoning Analogy Proceduralization

Figure 50. Jace's Narratives for Solving -a + □ = b (a, b > 0 and b > a).

Routines

Jace's routines transitioned across the session (see Figure 51). He transitioned from utilizing thinking that was less typically utilized in his sessions to drawing upon thinking that he used frequently. Similarly, Jace did not use number lines that frequently (25% in Session 1; 13% in Session 2) and transitioned to writing vertical or horizontal number sentences, which he drew frequently (92% of the time in both Sessions 3 and 4). Changing from drawing numbers lines, which he did not do often, to writing number sentences, which he did do frequently, may point to Jace becoming more familiar with this particular open number sentence type.

Individual Session 1 -4 + □ = 10	Individual Session 2 -6 + □ = 15	Individual Session 3 -3 + □ = 14	Individual Session 4 -4 + □ = 10
In this session, Jace used a number line in 5 of the 20 open number sentences, or 25% of the time. In this session Jace used Translation 30% of the time.	In this session, Jace used a number line in 3 of the 23 open number sentences, or 13% of the time. In this session, Jace used Algebraic Reasoning 26% of the time. Jace used Translation 22% of the time.	In this session, Jace used vertical number sentences in 23 of the 25 open number sentences, or 92% of the time. In this session, Jace used Algebraic Reasoning 40% of the time.	In this session, Jace used horizontal number sentences in 23 of the 25 open number sentences, or 92% of the time. In this session, Jace used Algebraic Reasoning 32% of the time. He used Analogy 68% of the time. He used Proceduralization 60% of the time.

Figure 51. Jace's Routines for Solving -a + □ = b (a, b > 0 and b > a).

Conjecture about the Influence of the Group Sessions on Jace's Learning

As mentioned earlier in this chapter, Jace preferred Proceduralization. Although the Group Sessions did not explicitly promote the learning of Proceduralization, the experiences in the Group Session provided Jace the opportunity to become more comfortable with this problem type. Jace transitioned from using less efficient strategies to more efficient strategies. For example, he transition from using a number line drawing with three distances to a number line with two different distances. Then, he transitioned using a number line to not drawing one at all in the last two sessions. He transitioned from explaining his use of commutative property in Session 3 to utilizing a well-established "procedure" in Session 4 for this problem type.

Describing the Learning of -a – b = □ (a, b > 0 and a > b)

For all four sessions, Jace solved the problem type -a – b = □ (a, b > 0 and a > b) incorrectly. That is, in Session 1, Jace solved -5 – 4 = □ incorrectly by answering -1 instead of -9 (see Figure 35). In Session 2, Jace solved -9 – 8 = □ incorrectly by answering -1 instead of -17. In Session 3, Jace solved -5 – 3 = □ solved incorrectly by answering -2 instead of -8. In Session 4, Jace solved -5 – 4 = □ incorrectly by answering -1, and then changing his answer to 9. Although he answered this problem type incorrectly across the sessions, Jace was still learning. For example, how Jace solved this and what CMIAS he used varied across the sessions (see Figures 52, 53, 54, and 55).

Word Use

Jace's word use changed across the sessions (see Figure 52). For Session 1 and 2, Jace's word use was focused on the role of the minus sign in the problem. In Session 1, Jace stated that the minus sign could be "drag[ed]" to a different location. In Session 2, Jace's words were still focused on the signs. By Session 3, Jace focused on the magnitude of the numbers with different signs. Although Jace did not mention dragging a minus symbol to a different location like he did in Session 1 or changing a minus sign to a plus sign, Jace did treat -5 – 3 as treated -5 + 3 in this same session. In the last session, Jace again started to think -5 – 4 was -1, as he typically thought. However, he considered changing his mind when he compared -5 – 4 to 4 – -5. Jace changed his mind to that -5 – 4 would equal 9. Although this is not correct, it does represent a change in his thinking and getting closer to the correct answer.

Individual Session 1 -5 – 4 = □	Individual Session 2 -9 – 8 = □	Individual Session 3 -5 – 3 = □	Individual Session 4 -5 – 4 = □
J: Ok, I can do this way. It's ... Because it's subtraction, you don't have to keep this (crosses off the "-" of -5). But, never, to make it easier, you would have keep it right away. So five minus four would equal one, but that would be negative one you drag this over here (points at -5, then draws a connecting line from -5 to box). Five minus four is one.	J: (Writes a vertical problem.) I think the answer would be negative one because ... because nine minus eight would equal one. But, it's like doing negative nine plus eight, if you change that (changes the minus sign to a plus sign) I guess. And that would equal negative one, would be the way that I think so...	J: (Draws a vertical problem. -5 - 3 = -2) I think the answer would be negative two because the negative number is bigger than the one that's not a negative number, so your answer would still be negative. T: Ok. So ...Can you explain how you were thinking about that? J: Like ... This number is the negative number (points at -5 and uses marker to circle it) and this number is the regular number (circles 3 with marker). Since this number (points at -5) is the negative number and this one's not (points at 3), it's still going to be in the negative number because this one is bigger (points at -5). T: Ok. So you think the answer's .... J: Negative two. T: Ok. (Turns page.) What about this one? One minus negative three ...	J: (Thinks about this silently.) T: So what are the thoughts going through your head? J: Like I'm thinking maybe ... I'll write it down, but I think that negative five minus four (start drawing horizontal problem) would equal negative one. Because, just like last problem, it's a negative number and a subtraction. So, but, I don't know. T: What's got you kind of questioning yourself right now? J: Because... ah maybe not. You know what... I think it's nine because ...if you have a negative five and you flip the problem around. So four minus negative five that would be five because you are taking away a negative number from the four even though you don't have a negative number. So, it would be plus instead of minus a negative. I'll wait a second. T: Alright so you first thought it was negative one. J: mmm-hmm.

Figure Continues

			T: And, now you don't think it is anymore? What made you think that it's not that anymore? J: Because it's ... to be negative one it would have to be negative five minus negative four, because five minus four equals one and then they're all negative numbers. But, since it's not then it's going to be a different answer. T: Ok. So then you wrote four minus negative five equals nine (points at second horizontal problem). J: Mmm-hmm. T: How come you switched the order (points at 4 - -5)? J: Because I think it helped me understand it better. T: You think it helped you understand it better. Ok. J: At first it just looked like ... what is the problem, you know? But, then if you flip it around it just kind of makes sense.
Word use focused on the role of the minus sign in the open number sentence.	Word use focused on the role of the minus sign in the open number sentence	Word use focused on the magnitude of the numbers with different signs.	Word use focused on procedures. Word use illustrated that Jace was unsure and he began to make comparisons to other number sentences he knew.

Figure 52. Jace's Word Use for Solving -a – b = □ (a, b > 0 and a > b).

Visual Mediators

Jace's visual mediators changed across the sessions (see Figure 53). In the first session, Jace did not write a number sentence. Rather he drew a line from the -5 to the box, to indicate that the minus sign could be unattached from the negative five and reattached elsewhere. In Session 2, Jace drew a vertical number sentence for -9 – 8; however, he also changed the minus sign to a plus sign to the original horizontal number sentence. In Session 3, Jace did not change the minus symbol in the original horizontal number sentence; but, Jace did write a different vertical number sentence, -5 + -3 = -2. In the last session, Jace first wrote the horizontal number sentence -5 – 4 = -1. He then wrote a second horizontal number sentence, 4 – -5 = 9. This number sentence ended up changing his mind about -5 – 4 = □.

Individual Session 1 -5 – 4 = □	Individual Session 2 -9 – 8 = □	Individual Session 3 -5 – 3 = □	Individual Session 4 -5 – 4 = □
-5 – 4 = $\boxed{-1}$	-9 + 8 = $\boxed{-1}$ -9 - 8 -1	-5 – 3 = □ + -5 -3 -2	-5 – 4 = $\boxed{9}$ -5-4=-1 4- -5=9
Connecting line to illustrate the "drag" of the minus sign	Vertical Number Sentence Cross off of minus sign	Vertical Number Sentence	Multiple Horizontal Number Sentences

Figure 53. Jace's Visual Mediators for Solving -a – b = □ (a, b > 0 and a > b).

Narratives

Jace's Narratives changed across the sessions (see Figure 54). Jace used Proceduralization in all of the sessions, but Session 3. In Session 3, Jace seemed to draw upon Counterbalance, which he did not use frequently in this session or other sessions. Jace also used Analogy in Session 2 and 4. Jace also began using Algebraic Reasoning with this number sentence in Session 4. Jace seemed to consistently use Proceduralization with this problem type, but also tried to use other CMIAS with it.

Individual Session 1 -5 – 4 = □	Individual Session 2 -9 – 8 = □	Individual Session 3 -5 – 3 = □	Individual Session 4 -5 – 4 = □
Proceduralization	Analogy Proceduralization	Counterbalance	Proceduralization Algebraic Reasoning Analogy

Figure 54. Jace's Narratives for Solving -a – b = □ (a, b > 0 and a > b).

Routines

Jace’s routines changed across the sessions (see Figure 55). Jace made use of typical routines in his visual mediators in Sessions 2, 3, and 4. Typically, Jace used vertical or horizontal number sentences. Jace also made use of typical routines with his CMIAS in Sessions 1, 2, and 4. Jace’s most utilized CMIAS were Proceduralization, Analogy, and Algebraic Reasoning, which he used in these sessions. However, in Session 3, Jace used Counterbalance, which was less routine for him.

Individual Session 1 -5 – 4 = □	Individual Session 2 -9 – 8 = □	Individual Session 3 -5 – 3 = □	Individual Session 4 -5 – 4 = □
In this session, Jace used connecting lines to numbers in 3 of the 20 open number sentences, or 15% of the time.	In this session, Jace used vertical number sentences in 19 of the 23 open number sentences, or 83% of the time.	In this session, Jace used vertical number sentences in 23 of the 25 open number sentences, or 92% of the time.	In this session, Jace used horizontal number sentences in 23 of the 25 open number sentences, or 92% of the time.
In this session Jace used Proceduralization 65% of the time.	In this session, Jace used Analogy 48% of the time and Proceduralization 70% of the time.	In this session, Jace used Counterbalance 24% of the time.	In this session, Jace used Proceduralization 60% of the time, Algebraic Reasoning 32 % of the time, and Analogy 68% of the time.

Figure 55. Jace’s Routines for Solving -a – b = □ (a, b > 0 and a > b).

Conjecture about the Influence of the Group Sessions on Jace’s Learning

This problem type was notoriously challenging for Jace and the other students. Jace used Proceduralization until Individual Session 3. Notably, Individual Session 3 was Jace’s highest use of Counterbalance and came after the Group Sessions where Counterbalance was being promoted through the use of contexts and a game. Perhaps Jace used Counterbalance here as a way to think about a challenging problem type differently; however, his Proceduralization was well established prior to this and his use of Counterbalance here only supported his “wrong” answer. However, in the last session, Jace felt unsure about this problem type. He changed his answer and he made an Integer Analogy, comparing -5 – 4 to 4 – -5. Group Sessions 6 and 7, although promoting Translation and Relativity, incorporated these problem types. Although he did not use Translation or Relativity here, it's possible that since his "wrong" Proceduralization was broke down in Group Session 6 and 7 that Jace decided to question his thinking with this problem type.

Describing the Learning of -a – □ = -b (a, b > 0 and b > a)

For three sessions, Jace did not solve the problem type -a – □ = -b (a, b > 0 and b > a) correctly. However, by the last session Jace solved problem type -a – □ = -b (a, b > 0 and b > a) correctly. That is, in Session 1, Jace solved -12 – □ = -13 incorrectly by answering -1 (see Figures 49). In Session 2, Jace solved -15 – □ = -16 incorrectly by answering -1. In Session 3, Jace was unsure how to solve -10 – □ = -11. In Session 4, Jace solved -12 – □ = -13 correctly by answering 1. Although Jace eventually was able to answer this problem type correct, Figures 56, 57, 58, and 59 illustrate Jace's learning of problem type -a – □ = -b (a, b > 0 and b > a) by including Jace's transcripts (word use), drawings (visual mediators), CMIAS use (narratives), and describing how much Jace used that narrative and type of visual mediator in that particular Individual Session (routines).

Word Use

Jace's word use changed across the sessions (see Figure 56). In Session 1, Jace's word use was focused on the steps he did to solve the problem and on the minus sign. In Session 2, Jace again focused on describing how to solve the problem; however, this time, Jace did not focus on the use of the minus symbols. In Session 3, Jace compared

-10 – □ = -11 to -10 – 4 = -6.

Individual Session 1 -12 – □ = -13	Individual Session 2 -15 – □ = -16	Individual Session 3 -10 – □ = -11	Individual Session 4 -12 – □ = -13
J: That just throws off what I just said, kind of. Because, I said that no matter what you do, when, like add or subtract, you will get... Like the value of this number (points at -12) will go down. But now you have subtraction and the total (points at -13) is bigger than what you started out with (points at -12) even thought it's subtraction. So, it's kind of confusing. But I think that it would be negative one. Twelve minus negative one would equal negative thirteen because twelve plus one is thirteen (Writes -1 in box). T: So you said that this problem kind of conflicts with what you were saying here (points a blank paper indicating previous problem). Can you describe how that conflicts over here? J: I was saying that you don't need the negative symbols. But, I guess from this problem (points to -12 - ? = -13), that you do need the negative symbols even right here. And the, if you are doing adding and subtracting, it does matter. It's either going to go up or go down (motions with hand up and down). T: Alright, what do you mean go up or go down? Are you talking about movements or numbers? J: Numbers. Do you see right here (points at number line) the value went down. I guess if you had, like a different, it could go up (and motions to the left). Like I was thinking that no matter what it would go down (motions to the right).	J: (Draws a vertical problem.) I think (writes -1 in the box). I think it would be one because fifteen plus one equals sixteen. If you did negative fifteen minus one, negative one. That would be negative sixteen. Because you have two negatives right there (points at -15 and -1) and you get (points at -16) another negative.	J: Mmm. I'm not really sure about this one either. Because if you negative ten minus negative one, then that would get you negative nine. T: mmm-hmm. J: I'm not sure how to ... It's kind of hard to describe because if you subtract a whole number, like let's say you did negative ten minus four, that would get you negative six because four is a whole number. T: Ok. I think what you started off doing was really productive. You started off saying negative ten minus negative one would be negative nine. You told me that you know it's not that. Ok ... can you think of any other reasoning like that might help you? Or ... J: No, not really. I really don't know about this problem.	J: (Draws horizontal problem.) Wait. I'm confused. Because on the last problem. I said it was just like subtracting regular numbers. But now you are in the negative numbers and you can't go any deeper, any lower than a negative number. So like you can't do negative twelve minus negative one equals negative thirteen. That would be negative eleven. T: Ok. J: And, it can't be negative twelve ... well, maybe it is. It's just one because you are going down. So, negative twelve, negative thirteen. T: Ok. So you think the answer is … J: One. T: Positive one? J: (Nods yes.)
Word use focused on the steps he did to solve the problem and on the minus sign.	Word use focused on describing how to solve the problem; however, Jace did not focus on the use of the minus symbols.	Word use focused on comparing the open number sentence to a different number sentence. Jace then expressed that he didn't know how to do this problem.	Word use illustrated that this problem was confusing. Jace stated what he couldn't do and transitioned to reasoning about "going down" and concluded with the correct answer.

Figure 56. Jace's Word Use for Solving -a – □ = -b (a, b > 0 and b > a).

Jace then expressed that he did not know how to do this problem. In Session 4, Jace began with expressing that this problem was confusing. He then stated what he couldn't do and transitioned to reasoning about "going down" and concluded with the correct answer

Visual Mediators

Jace's visual mediators changed across the sessions (see Figure 57). In Session 1 the only visual mediator that Jace produced for this problem was the number in the box. Then, in Session 2, Jace drew a vertical number sentence. The operations were crossed off in it, as though he couldn't decide between using a "+" or a "-." In Session 3, Jace did not draw anything or provide an answer in the box. Although Jace first wrote the horizontal number sentence, -12 – -1 = -12, he wrote just 1 in the box and stated that the answer was 1.

Individual Session 1 -12 – □ = -13	Individual Session 2 -15 – □ = -16	Individual Session 3 -10 – □ = -11	Individual Session 4 -12 – □ = -13
-12 – $\boxed{-1}$ = -13	-15 – $\boxed{-1}$ = -16 -15 ~~+~~ -1 -16	-10 – □ = -11	-12 – $\boxed{1}$ = -13 -12--1=-13
Answer in Box Only	Vertical Number Sentence	Blank	Horizontal Number Sentence

Figure 57. Jace's Visual Mediators for Solving -a – □ = -b (a, b > 0 and b > a).

Narratives

Jace's narratives changed across the sessions (see Figure 58). In Session 1, Jace used Analogy, Proceduralization, and Translation. Then, in Session 2, Jace used Analogy and Proceduralization, not drawing upon Translation. By Session 3, Jaced used only

Analogy. However, in Session 4, when Jace got the problem correct, he used Proceduralization and Translation.

Individual Session 1 -12 – □ = -13	Individual Session 2 -15 – □ = -16	Individual Session 3 -10 – □ = -11	Individual Session 4 -12 – □ = -13
Analogy Proceduralization Translation	Analogy Proceduralization	Analogy	Proceduralization Translation

Figure 58. Jace's Narratives for Solving -a – □ = -b (a, b > 0 and b > a).

Routines

Jace's routines changed across the sessions (see Figure 59). When solving this open number sentence type, Jace drew upon things he used routinely and non-routine things as well. In both Sessions 1 and 3, Jace used visual mediators that were not typical for him. For example, in Session 3, this was the only problem he left blank and did not draw anything. However, in Session 2 and Session 4, Jace produced visual mediators for this that were typical for him. That is, he typically drew both vertical and horizontal number sentences.

Individual Session 1 -12 – □ = -13	Individual Session 2 -15 – □ = -16	Individual Session 3 -10 – □ = -11	Individual Session 4 -12 – □ = -13
In this session, Jace wrote his answer in the box with no other drawing. He did this with 3 of the 20 open number sentences, or 15% of the time. In this session, Jace used Analogy 50% of the time, Proceduralization 65% of the time, and Translation 30% of the time.	In this session, Jace used vertical number sentences in 19 of the 23 open number sentences, or 83% of the time. In this session, Jace used 48% of the time and Proceduralization 70% of the time.	In this session, Jace this is the only number sentence that Jace left blank out of 25 open number sentences, or 4% of the time. In this session, Jace used Analogy 52% of the time.	In this session, Jace used horizontal number sentences in 23 of the 25 open number sentences, or 92% of the time. In this session, Jace used Proceduralization 60% of the time and Translation 28% of the time.

Figure 59. Jace's Routines for Solving -a – □ = -b (a, b > 0 and b > a).

For all four sessions, Jace drew upon CMIAS that used often. For example, he used Proceduralization in Session 1, 2, and 4 and he used Analogy in Sessions 1, 2, and 3. However, in Sessions 1 and 4, Jace used Translation, which was one of the CMIAS that he used less frequently. Jace utilized Translation when he got the problem correct in the last session.

Conjecture about the Influence of the Group Sessions on Jace's Learning

Again, this was also a problem type that was notoriously challenging for Jace. Although aware that it was a challenging problem type, he typically continued to draw upon Proceduralization to solve the problem. Perhaps he felt uncomfortable with this problem type and comfortable with Proceduralization. However, using Translation paired with Proceduralization he got the answer correct in the last session. The timing of this Translation use is notable. It comes after Jace pariticipated in Group Sessions that promoted Translation and incorporated this problem type. Although he did not reference this experience, the timing of Translation and a "correct" answer is noteworthy.

Summary

Learning is defined as a change in mathematical discourse. Thus, identifying changes in CMIAS use is a way of describing learning. This chapter began with discussion of Jace's learning of use his dominant CMIASs, like Proceduralization and Analogy. Jace learned to use CMIAS that he did not previously use, like Counterbalance. Jace learned to use his prominent CMIAS, Proceduralization and Analogy, more efficiently. Then this chapter zoomed in to describe the learning of specific problem types. Jace's learning was described across three open number sentence types. Whether

Jace was getting the problem “correct” or “incorrect” he was learning, illustrating changes in his thinking.

CHAPTER VI

DISCUSSION, EDUCATIONAL RECOMMENDATIONS, & FUTURE RESEARCH

This chapter opens with a discussion about the CMIAS in relationship to the research question, focusing on the use and learning of CMIAS. Then, this chapter makes explicit connections from the CMIAS to both the WoR (Bishop et al., 2014a) and Mental Models (Bofferding, 2014). Educational recommendations are highlighted and connected to the discussion points. Then, this chapter concludes with future research suggestions.

Discussion about the Refinement & Use of CMIAS

Discussion about the use of the CMIAS will be shared next about the following two points: (a) The students prominently used certain CMIAS, even though the students were flexible in their use of the CMIAS; and, (b) Students also utilized the CMIAS differently when posing stories and when solving open number sentences.

Prominence & Flexibility in Use

The students in this study eventually demonstrated use of nearly all of the CMIAS. Jace used all of the CMIAS at some point in the study. Alice and Kim used all of the CMIAS except Relativity. Although the students were capable and utilized the majority of the CMIAS, the students demonstrated a preference for certain CMIAS. For example, Alice used Bookkeeping more than Jace and Kim used Bookkeeping. Jace and Kim used Proceduralization more than Alice used Proceduralization. The students demonstrated that they might have preference to utilizing some CMIAS over others.

Also, students often employed use of several CMIAS when solving a singular problem. Although students may have had prominent CMIAS, the students were also flexible in their use of the CMIAS.

Different Uses in Contextual & Symbolic Problems

The students used different CMIAS when they solved open number sentences then they did when they posed stories. For example, as Alice generated stories for integer open number sentences for addition and subtraction, she used only Bookkeeping. Yet, Alice used Bookkeeping, Counterbalance, Translation, Analogy, Algebraic Reasoning, and Proceduralization when she solved integer open number sentences. Although Jace used more than just Bookkeeping to generate stories for integer open number sentences, he still only utilized Bookkeeping, Translation, and Relativity. Yet, when Jace solved open number sentences he used Bookkeeping, Counterbalance, Translation, Relativity, Analogy, Algebraic Reasoning, and Proceduralization. It seems that student use different CMIAS in different situations. For example, while both Alice, Kim, and Jace prominently employed Bookkeeping to pose stories for open number sentences, other CMIAS were more prominent for them as they solved open number sentences. Because the students used a greater variety of CMIAS when solving open number sentences, the students' use of the CMIAS was also more flexible as they solved open number sentences.

Discussion about Learning & CMIAS

Discussion about the learning and the CMIAS will be focused on the following two points: (a) Although learning to operate with the integers is important, the CMIAS mark a description of what it means to learn integer addition and subtraction that is beyond operations; and, (b) The students' use of the CMIAS changed over time.

Beyond Operations

The focus on school instruction and often research is on the teaching and learning of integer operations. Learning about the integers is more than just operating with the integers (Kilhamn, 2009; Lamb et al., 2013) and more than just getting answers right or wrong. The CMIAS represent a way to describe learning about integer addition and subtraction that is beyond operations. Students learn more than just operations; students learn to use various CMIAS and become flexible in use with the CMIAS. The students were using the different CMIAS even when they were getting problems incorrect. As they changed the CMIAS that they used, they were learning.

The learning of integer addition and subtraction changed over time, more than just the just the correctness. Of course, operating correctly matters. But, Jace demonstrated that students not only solve more problems correctly over time, but that students' learning of integer addition and subtraction entails much more than just correct answers. The changes in Jace's word use, visual mediators, narratives, and routines are descriptions of learning that are more robust than correctness. Jace used different words, he drew different things, used the CMIAS differently, and his repetitive behaviors changed. This was productive learning and took significant time to develop.

The students in this study invented robust, productive ways of reasoning that were often unexpected. Recall from Chapter IV, when Alice drew 20 boxes to represent -20 and crossed off 15 boxes to represented adding 15. In that chapter, Alice's actions were described as representing -20 – -15 to solve -20 + 15. Students are capable of solving problems with negative integers in unique, unconventional ways that are both sophisticated, containing important mathematical ideas beyond operations. Using

Translation means making use of mathematical ideas like vectors and distance. Using Counterbalance means making use of mathematical of neutralizing quantities.

Conceptual Change

Learning was described in this study as identifying changes in mathematical discourses. The narratives, which were described by Jace's use of the CMIAS, changed over time. These changes in CMIAS represent an illustration of conceptual change. The conceptual changes in use of CMIAS marks an illustration of development often missing in the current literature. Bofferding (2014) used the Mental Models paired with pre- and post-tests around an instructional intervention to capture conceptual change that occurred. As a field, we need more descriptions of these types of development perspectives, particularly if we interpret the negative integers as secondary intuitions (Fischbein, 1987). If negative integers are interpreted as a secondary intuition, then conceptual change can not be measured with cross-sectional studies. Rather, conceptual change needs to be captured around instructional interventions. Additionally, if we consider conceptual change as learning, then we must use CMIAS or Mental Models or other descriptors as ways to describe learning.

Affordances & Limitations of CMIAS

Each of the CMIAS have hindrances and affordances for student learning, similar to how pedagogical models have hindrances and affordances (Vig, Murray, & Star, 2014). By stating that the CMIAS have limitations, this refers to the drawbacks or misconceptions that may be created if students use only one CMIAS or do not extend their thinking within a particular CMIAS. For example, if a student only uses counting strategies with Translation and never extends their thinking with Translation to incorporate use of distance on a number line, there can be a confounding between tic marks and spaces on the number line, thus limiting the students' thinking (e.g., Barrett et al., 2012). Or, if students strictly reason with Analogy, with only whole number analogies, he or she might think that $1 - \square = 3$ is impossible. If students use only Translation, they lose opportunities to build off of more quantitative thinking, like with Bookkeeping and Counterbalance. Yet, if students only used Bookkeeping, they would lack opportunity to think about integer addition and subtraction with movement like they do with Translation.

One of the affordances of the CMIAS is that they represent the thinking and mathematical uses of the integers that are more than just operations. Imbedded in each of the CMIAS are mathematical ideas of operations with order and magnitude a silent partner. Some of the CMIAS seem to support order more; whereas, other CMIAS seem to support magnitude more. For example, Bookkeeping seems to support magnitude more than order; yet, order is also an attribute of Bookkeeping. Magnitude seems more prominent than order in Bookkeeping because conceptualizing a gain of 17 as larger than a gain of 15 or even a loss of 12 is quite common. Yet, when ordering a loss of 20 and a gain of 5, one might reason that 20 is bigger because it has a greater "loss," which is magnitude based reasoning. Yet, with Translation order seems supported more than magnitude. Although students discussed moving "deeper," which is directed magnitude, the students would determine their answer based on order, through the use of a number path or number line.

Both an affordance and hindrance of the CMIAS is that some seem supported by contexts (i.e., Bookkeeping, Counterbalance, Translation, Relativity) and others seem supported by solving open number sentences (i.e., Proceduralization, Algebraic Reasoning, Analogy). Some appear to be utilized in both contexts and number sentences (i.e., Bookkeeping, Counterbalance, Translation). While others were utilized only when generate contexts (i.e., Relativity) or when solving open number sentences (i.e., Algebraic Reasoning, Proceduralization). When students use only one or two CMIAS with contexts this is a hindrance because using one or two CMIAS is not enough to fully understand the addition and subtraction of integers. However, when students solve open number sentences freely they use many CMIAS and this is an affordance because the use of a variety of CMIAS provides a robust understanding of integer addition and subtraction. Yet, learning to both generate contexts and solve open number sentences are important to learning integer operations.

Connecting the CMIAS to Other Research Agendas

Bofferding, Wessman-Enzinger, Gallardo, Salinas, & Peled (2014) made an explicit call to "build bridges" between integer research and agendas. One of the goals of this entire dissertation, explicitly discussed in Chapter II, was to connect the existing literature to the CMIAS. This last chapter will conclude with connecting the modified CMIAS to recent research: Mental Models (Bofferding, 2014) and Ways of Reasoning ([WoR], Bishop et al., 2014a). The discussion below is my interpretation of others' work and how it connects to my own in an effort to build the bridges between the agendas. If we can identify places of similarities and differences, then we will learn more about our own research and our future research with student thinking about integer addition and subtraction.

CMIAS & WoR

Both the CMIAS and WoR are broad ways of describing thinking, larger than a strategy for solving an integer addition or subtraction problems itself. WoR focus on broad ways of reasoning; whereas, the CMIAS are representing of a model of thinking, which is only a singular tenet of describing thinking, that encapsulates mathematical uses of the integers. Yet, the CMIAS and WoR seem intimately related. Pointing out similarities and differences will only help us understand student thinking about integer and subtraction better.

Order is a WoR that describes students' use of integers that is sequentially based reasoning, which includes counting strategies, motion, and movement. This is related to both Translation and Relativity. "Counting strategies, motion, and movement" which are descriptors of the WoR Order is nearly an isomorphic description of Translation. Order is an important component to using Relativity since it is an ordered comparison. However, motion and movement are directed magnitudes and vectors, which are discussed in the WoR description of Magnitude. The WoR description of magnitude includes students' use of cardinality with the integers. This way of reasoning includes both contextual comparisons to debts and assets and directed magnitudes, or vectors. The use of directed magnitudes or vectors relates to Translation. However, with the CMIAS students used Counterbalance with magnitude that was not explicated directed in this study. There is directed magnitude and absolute magnitude. That is, directed magnitude is magnitude that has a direction, like a vector from 0 to 2. There is absolute magnitude that is magnitude without an explicit direction, like an undirected distance from 0 to 2. When students, like Alice and Jace, used Counterbalance, they would reason that problems like

12 + -16 = -4 because “16 is more than 12.” This is an example of absolute magnitude. Certainly, directed magnitude could be utilized here, but it is not explicit in the statement. When students in this study used directed magnitude, or Translation, they often talked about moving from one direction to another.

Logical necessity and formalisms is a WoR that describes students’ use of structural similarities about problems and generalizations to solve the problems. This WoR is related to Algebraic Reasoning, Proceduralization and Analogy. Students used algebraic properties, like the commutative property of addition and addition. Students also changed the structure of number sentences with Algebraic Reasoning to solve problems. With Proceduralization, students created and used generalizations to solve problem. With Analogy, Students made comparisons, or analogies, from problem type to another (e.g., comparing -2 + -3 to 2 + 3). However, Analogy is not explicit in the WoR, although it could possibility be included in the Logical Necessity and Formalisms because of the “structural similarities between problems”; however, Analogy was sometimes employed when the problem were not necessarily structurally similar, a logical equivalent, or a formalism of mathematics. For example, Jace compared -5 – 4 to 4 – -5, which is not mathematically equivalent, but was a productive Analogy for him.

Computation is a WoR that describes students’ use of computations, rules, or procedures. This is directly related to the Proceduralization. With the CMIAS, students creating rules and developing generalizations are considered similar ways of thinking. With WoR, creating rule and developing generalizations are considered different ways of thinking. Thus, the generalization and rules of Proceduralization seem to overlap both the WoR Formalisms and Computation.

Table 24

Relating the CMIAS to the WoR from Bishop et al. (2014a)

Ways of Reasoning (WoR)	Relating the CMIAS to the WoR
Order	Students use sequentially based reasoning, which includes counting strategies, motion, and movement. This seems related to both *Translation* and *Relativity*. Motion and movement are related to directed magnitudes and vectors, which are discussed below with Magnitude.
Magnitude	Students use cardinality. This way of reasoning includes both contextual comparisons to debts and assets and directed magnitudes, or vectors. The use of directed magnitudes or vectors relates to *Translation*. However, students used *Counterbalance* with undirected magnitude in this study.
Logical necessity and formalisms	Students use structural similarities about problems and generalizations to solve the problems. This seems related to *Algebraic Reasoning*, *Proceduralization* and *Analogy*. Students used algebraic properties like commutativity and inverse, and changing the structure of number sentences with *Algebraic Reasoning* to solve problems. Students created generalizations with *Proceduralization* to solve problems. Students make comparisons and analogies between problem types. When students make these comparisons they are using *Analogy*. Analogy is not explicit in the WoR, although it could possibility be included in the Logical Necessity and Formalisms; however, *Analogy* is not necessarily logically equivalent or a formalism of mathematics.
Computation	Students use computations, rules, or procedures. This is seems directed related to the *Proceduralization.* With the CMIAS*,* students created rules and developed generalizations are considered similar ways of thinking. The generalization and rules of *Proceduralization* seems to overlap with the WoR Formalisms.
Limited	Students illustrate thinking incomplete or limited views of negative numbers. There is not a CMIAS to compare this to because any of the thinking with in the various CMIAS can be limited. For example, *Analogy* would be limited if student used only whole number analogies and never integer analogies.

Limited is a WoR that describes students' incomplete thinking or limited views of negative numbers. There is not a CMIAS to compare this to because any of the thinking with in the various CMIAS can be limited. For example, Analogy would be limited if students used only Whole Number Analogies and Never Integer Analogies. Or, Translation would be limited if students used only counting on a number path and never used directed distance on a number line. Table 24 summarizes the relationships between the refined CMIAS and WoR described above.

CMIAS & Mental Models

Both the CMIAS and Mental Models are intimately related to cognition and conceptual change. Neither the CMIAS nor Mental Models claim to be descriptions of "mental images" or "visualizations" that children or students have; rather, both the CMIAS and Mental Models are researcher tools and descriptions that attempt to best describe the thinking that students demonstrate. The CMIAS provide broad descriptions of thinking and mathematical use of integer addition and subtraction; the Mental Models provide specificity of thinking about value, order, and directed magnitude, which are important components to understanding addition and subtraction. A major instructional implication of Bofferding's (2014) research supporting Mental Models is to promote conceptual change in students. Both the CMIAS and Mental Models are descriptions of student thinking with implications for promoting conceptual change where both the CMIAS and Mental Models are tools to document these conceptual changes.

Imbedded within the descriptions of the Translation and Counterbalance are descriptions of students use of magnitude. Using Translation, students would use directed magnitude as they moved across the number line and documented this directed distance.

Similarly, using Translation, students utilized absolute magnitude when students examined distance with no explicit description or use of direction. Furthermore, students demonstrated use of absolute magnitude when they used Counterbalance and compared the absolute magnitude of two numbers to compute an addition problem, like 12 + -16. The results of this dissertation study point to the proposal of the development of a third set of Mental Models. Bofferding (2014) described two sets of Mental Models: Value and Order, and Directed Magnitude. Perhaps, students may also utilize a set of Mental Models for Absolute Magnitude, an addition to Bofferding's Mental Models.

The Mental Models may also represent important mathematical thinking that is imbedded in each of CMIAS. Although each of the Mental Models is needed to think about the integers, students may utilize some Mental Models more dominantly than others. Figure 53 below illustrates the hypothesized relationship of each of the CMIAS with Bofferding's (2014) Mental Models, as well as, the proposed Absolute Magnitude Mental Model. In Figure A some of the arrows that represent Mental Models are larger than other arrows. This represents a hypothesis that some Mental Models may be more dominate or influence the CMIAS in different ways. For example, if a student draws dominantly upon an Absolute Magnitude Mental Model, then he or she may use Bookkeeping or Counterbalance more. If a student draws dominantly upon Value, Order, and Directed Magnitude Mental Models, then he or she may use Translation or Relativity. This is not to say that Absolute Magnitude is not an important component to utilizing Translation or that Directed Magnitude is not an important component to utilizing Bookkeeping. Rather, this is saying that the Mental Models may take time to develop when using each of the CMIAS. Furthermore, the Mental Models and CMIAS may be synergistically related and dependent upon each other.

Figure 60. Relating the CMIAS to the Mental Models from Bofferding (2014).

Figure 60 provides a graphic that illustrates the conjectured relationship between the Mental Models and the CMIAS. For example, as part of Figure 60, there is a large circle representing the CMIAS Bookkeeping and Counterbalance. Within in this circle are three Mental Models. One of the Mental Models, the Absolute Magntidue Mental Model, is larger in this graphic suggesting that this Mental Model may be more prominent than the other Mental Models when drawing upon Bookkeeping and Counterbalance.

Significance of Results

The refined descriptions of the CMIAS extend the literature in several ways. First, the CMIAS present descriptions of student thinking about integers that are developed from students' responses in both contextual and symbolic tasks. Prior to the development and descriptions of the refined CMIAS, descriptions of student thinking were situated dichotomously in either contextual or symbolic settings. The development of these from both posing stories and solving open number sentences is important because it shows a tighter way thinking between contextual and symbolic settings. Similarly, although students used them differently, the students also used the same CMIAS in both contextual and symbolic settings. Although there is different utilization, the relationship between the CMIAS used in contexts and symbolic settings illustrated a connection.

Second, the refined CMIAS present new conceptual models, such Analogy, that need to be defined in the research on student thinking about integers. For example, the WoR descriptions (Bishop et al., 2014a) do not have an explicit reference or definition to analogies and the refined CMIAS include this description. Although as a field we recognize that students use analogies to solve integer addition and subtraction problems (Bofferding, 2010; Human & Murray, 1987; Murray, 1985), there was lack of clarity about the definition. Also, an asset to definition of Analogy is the distinction between Whole Number Analogies and Integer Analogies. Although the literature refers to and supports the existence of Whole Number Analogies, this is the first distinction of Integer Analogies. Along this lines, when analogies have been discussed prior they are typically discussed with reference to a static analogy. Because Whole Number Analogies and Integer Analogies are distinguished in this results, this points to research in the future that captures the possible development from Whole Number Analogies to Integer Analogies.

Third, the descriptions of the refined CMIAS explicitly define the role of zero as well as the positive and negative integers within each of the CMIAS definitions. For example, zero represents as a state of neutralization in Counterbalance and zero represents a translation of no movement in Translation. Because zero and the positive and negative integers have defined roles here, it will be easier to design research, pedagogical tasks, and explore student thinking better with reference to the refined CMIAS. Similarly, it is also highlighted that zero and the integers do not have explicitly defined mathematical meanings in the CMIAS description, like Analogy and Proceduralization. This contribution of defined meanings of the integers and recognition that sometimes the integers do not have defined meanings is a major contribution. In addition to the current literature supporting each of the CMIAS, both teachers and researchers agree that we should use ideas of Counterbalance (e.g., cancellation models) and Translation (e.g., number line models); however, the CMIAS provide a tighter and more refined way of talking about all of these ways of mathematically thinking and using the integers, especially with the defined roles of the integers.

One of the contributions of the results of this study to the literature is that the student use of the CMIAS is that the student use is described in from Individual Sessions that included both contexts and open number sentence. The results of this study are some of the first attempts that have tried to explicitly make a connection between student thinking in symbolic settings and student thinking about contexts.

The descriptions of learning about addition and subtraction extend the literature as well. With integer addition and subtraction, learning is typically captured in the literature either with singular interviews or with pre- and post-assessments around an instructional intervention. These descriptions of learning happened over extended time, of 12-weeks, with four assessments give more detail to learning and development. Furthermore, the focus of learning has traditionally been centered on the correctness of operating with the integers and the description of learning here goes beyond just correct. Bofferding (2014) described learning as conceptual change; similarly, these results capture conceptual change of the CMIAS. Learning as described in this study extends the literature by including descriptions of change in word use, visual mediators, narratives, and routines. And, we know little about the word use and the types of visual mediators that children produce over time in the area of integer addition and subtraction.

Educational Recommendations

The CCSSO & NGA recommendations include all four operations in Grade 7. First, this study illustrates that students are capable of productively operating with integers at an earlier age than the CCSSO & NGA recommendations. However, this is consistent with the existing literature (e.g., Bishop et al., 2014a, 2014b; Bofferding, 2014; Bofferding & Wessman-Enzinger, 2015; Featherstone, 2000; Wessman-Enzinger & Bofferding, 2014). Second, learning about integers and integer addition and subtraction takes significant time. In this study, the students spent 12-weeks, 2 to 3 days a week, on only integer addition and subtraction. While the students were able to productively operate with the integers, there were also struggles and challenges throughout the weeks. For example, Jace solved some open number sentence problem types incorrectly for all of the 12-weeks. Similarly, some of the problem types he did not get correct until the last sessions. Although the CCSSO & NGA recommendations include all four operations in Grade 7, there are many other mathematical expectations that year as well. If we wish to support student thinking, development of invented strategies, and discussion and critique about these, as indicated by the Mathematical Practices within the CCSSO & NGA recommendations, it seems that more time needs to be given for the operations of the integers.

Current instruction and curriculum with integer operations is focused on operations with integers (i.e., symbolic uses with integers). The use context with the integers is often used as a pedagogical tool for the introduction of the integers (see, e.g., the argument of this presented in Chapter I) and contexts are also used as "applications" after instruction with integer operations. We need to re-think the role of what teaching integer addition and subtraction looks like. The results of this dissertation study highlight that students think about the integers differently when solving open number sentences and generating contexts for open number sentences. This suggests that we need to spend as much time teaching and learning the integers in contexts as we do with open number sentences to facilitate the development of these different ways of thinking. Similarly, we also need to re-conceptualize the role of context in integer operation instruction. That is, we should not always give students contexts and ask them use negative integers. Rather, we should give students contexts and ask the students what they would like to use (positive or negative integers) and support the use of both positive and negative integers. For example, for the problem of owing your friend 5 dollars and paying him back 2 dollars we can represent this situation correctly with both -5 + 2 and 5 – 2. Additionally, students should be generating contexts for number sentences more and not only solving problems in provided contexts. When students generate their own contexts, this provides insight into their thinking more and also provides a space for interesting discourse. For example, suppose students posed different stories for -5 + 2. One story may be, “He wanted 5 baseball card and received 2.” Another story may be, “He lost five pencils and got two more.” Productive discourse around these could be made about how these stories are similar. Also, productive discourse could focus on appropriate questions that go with these stories. For example, if I asked, “How many pencils do I have now?” for the latter story, that would not be appropriate for -5 + 2, even if it is a logical question.

Even when students are getting problems “correct,” they are still learning and developing their CMIAS. And, even when students are getting problems “incorrect,” they are still learning and developing CMIAS. For example, a description of Jace’s learning across one of the open number sentence types demonstrated that he solved a particular problem type correctly across the four Individual Open Number Sentence Sessions. Jace, although solving it correctly across all four Individual Sessions, produced different visual mediators, from empty number lines to number sentences. His word use changed, from discussion of movement to discussion about developed rules. Although he got the correct answer, he still demonstrated learning. Similarly, Jace solved one of the open number sentence types incorrectly across the four Individual Open Number Sentence Sessions. Although he solved this problem incorrectly, he demonstrated use of different CMIAS. He also changed his thinking and started to make analogies to other integer open number sentences, which could prove useful in the future. For example, for the first three session Jace reasoned that problem types like -5 – 4 = -1. In the last session, he reasoned that -5 –

4 might have equaled 9 because he compared -5 – 4 to 4 – -5 = 9. Although -5 – 4 is not equivalent to 4 – -5, it is a comparison that could prove to be productive and also helped him to break a challenging misconception that -5 – 4 = -1. Whether getting correct or incorrect answers, the thinking and learning about integer addition and subtraction was productive. Students need opportunities to both develop their current understandings and develop new ones. The focus of integer instructions should not be on obtaining the correct answers, but rather on fostering environments that support the creative invention of these various ways of thinking.

Summary of Educational Recommendations

The educational recommendations are summarized below. First, instruction for integers should be sooner than the CCSSO & NGA recommendations of Grade 7 and instruction with integer operations should be for a substantial amount of time, not just one school year. Second, students need to solve contextual problems, generate contexts, and solve open number sentences. Third, students should be supported to invent their own ways to solve integer addition and subtraction problems gives students opportunity for students build upon their prior understandings. Finally, facilitating students in discourse around these various ways to think about and use integers will give students opportunity for learning, or conceptual change. Of course, relating research to practice is important for all subjects, but it is especially important for topics that are notoriously challenging to teach (e.g., Piaget, 1948). Table 25 links the points made earlier in the discussion section about the CMIAS to the educational recommendations that were just discussed above.

Table 25

Connecting Discussion Points to Educational Recommendations

Discussion points about CMIAS	Educational Recommendations
Prominence & Flexibility in Use	Students need to solve contextual problems, generate contexts, and solve open number sentences by having time to invent their own ways to solve the problems in order to build upon their prior understandings.
Different Uses in Contextual & Symbolic Problems	We need to spend as much time teaching and learning the integers in contexts as we do with open number sentences to facilitate the development of these different ways of thinking. We need to re-conceptualize the role of context in integer operation instruction, having student generate contexts for number sentences more.
Beyond Operations	More time than recommended by the CCSSO & NGA needs to be allotted the teaching and learning of integers operations.
Conceptual Change	Allowing students to invent their own ways to solve integer addition and subtraction problems gives students opportunity for students build upon their prior understandings. Facilitating students in discourse around these various ways to think about and use integers will give students opportunity for conceptual change.

Limitations of the Study

Although an asset of this study is that extensive time was spent with three Grade 5 students, this is also the major limitation of this study. Because data was gathered only from a small sample of students, the results of this study are certainly influenced by the way that these three students reasoned; and, these students may not be an authentic representation of all fifth-grade students. However, generalizability can be made about these particular students and their ways of thinking in this study because of the extended time spent with the students.

Another limitation is the design and structure of the individual context sessions. It was challenging to determine what CMIAS the students were using when a CMIAS was promoted by a context, as well as, the students were only asked to write a number sentence and evaluate other number sentences. In the future, tasks like this need to be modified to incorporate more discussion, with less focus on writing a number sentence only, to understand student thinking better about these. Also, the context Individual Sessions did not include as many stories to generate for open number sentences as the open number sentence interviews had open number sentences for the students to solve. This is a limitation because the CMIAS are compared between the Individual Open Number Sentence Sessions and the Individual Context Sessions. Although it seems as if the students were using different CMIAS between the two, the students solved significantly more problems during the Individual Open Number Sentences Sessions. A challenge though was that the students did not appear to enjoy posing stories as much as solving the open number sentences so getting them to pose over twenty stories in a singular session was not possible.

In terms of making educational recommendations based on this study, one of the limitations is that this was not an authentic classroom. These students had specialized attention through both Individual and small Group Sessions. Similarly, the Group Sessions were guided by the student thinking, not driven by curricular expectations. Although the teaching experiment conducted lacks authenticity to a classroom, the teaching experiment methodology (Steffe & Thompson, 2000) was selected because of its close relationship to practice. Many methodologies used to investigate student thinking are not linked closely to practice; however, an affordance Steffe and Thompson (2000) methodology is its link to the practice of teaching with the teacher-researcher and witness acting as teachers who facilitate the thinking of the students, who are also researching the thinking that the students demonstrate.

Future Research

There is a wealth of future research that needs to be conducted in relation to the CMIAS and as a consequence of this dissertation study. First, more work needs to be done to understand how students demonstrate use of these CMIAS and learn to draw upon the CMIAS. For example, similar research using teaching experiment methodology could be conducted with more students, across various levels, and for a longer period of time. A larger scale study of this nature, with more students and across various levels, would provide further insight into the use of CMIAS and the learning of integer addition and subtraction. Also, this dissertation study demonstrated that students draw upon the CMAIS differently and the use of the CMIAS change over time. Work needs to be conducted that illustrates these different transitions of use with the various CMIAS. This type of study could be conducted by employing microgenetic methodology to intentionally capture these changes better.

As part of this study, the students drawings, or visual mediators, were examined. Although just implicitly reported on in this dissertation study, the drawings that the student produced did not follow the conventional curricular norms of number line or chip model use. For example, the drawings that Alice produced involved drawing boxes and tallies where the objects were crossed off or layered in unconventional ways (Wessman-Enzinger, in press). More work with the visual mediators (e.g., drawings, written number sentences) that the students produce, in relationship to the CMIAS, needs to be conducted. The ways that the students invented unique drawings to operate with the integers consequently highlights that we know little about the invented use of manipulatives that students may use when learning about integer addition and subtraction. Although we know how students use intuitively use manipulatives like Unifix to solve addition and subtraction problems with positive integers, we know very little about how students would use manipulatives, like chip models, intuitively. Vig, Murray, and Star (2014) highlighted that use of manipulatives, like chip models, have hindrances and affordances. Investigating students' intuitive use of manipulatives with integer addition and subtraction would provide further insight into the hindrances and affordances of the various pedagogical models typically utilized with integer operation instruction.

Admittedly, as a teacher-researcher I am cautious of using words like "learning progression" or "learning trajectory" because of the various connotations that are intentionally and unintentionally evoked. I am also fearful of using words that promote thinking that there is a singular path of learning or a singular hierarchical description of learning. However, it is undeniable that the students in this study did not use Relativity as abundantly as the other CMIAS. Yet, one of the students illustrated budding use of Relativity in the last individual context session when solving the last contextual problem about moving a tree (Wessman-Enzinger, 2015). Noting that Relativity is both historically and mathematically supported (e.g., Gallardo, 2000) and that the students did not use Relativity often in this study, it is possible that there may be learning progressions in relationship to the CMIAS that need to be identified. I purposefully used the plural use, rather than the singular use, of learning progressions. And, the influence of the group sessions on students' learning needs to be further examined. Alice demonstrated use of Bookkeeping and Counterbalance, more than Kim or Jace in this study. Similarly, Kim demonstrated use of Translation in this study more than Alice does. I conjecture that there are multiple learning progressions, where some students initially draw upon Bookkeeping and Counterbalance more and other students initially draw upon Translation more. I also conjecture that students will move towards Proceduralization, and that Relativity is a CMIAS that takes significant time to learn. Future work about what these possible learning progressions look like need investigation.

It is possible that certain open number sentence types promote different types of learning or CMIAS, it is important to study how the CMIAS are used differently across the various different open number sentence types. For example, it seems likely that problem types like -2 + -8 may evoke the use of Analogy and problem types like 12 + -16 may evoke use of Counterbalance or Proceduralization. Similarly, investigations into the orderings of the presentations of the various open number sentence types may influence how students solve integer problems and warrants further investigations. For example, if students first solve -2 – 5 then solve -2 – -5 second, this may influence the ways that students decide to solve -2 – -5. One of the things that was not explicitly reported on in this dissertation study, is that the students in this study were often extremely reflective and would want to compare how they solved problem types like -2 – 5 to problems like -2 – -5 even if the problems were not posed in that order or provided. If the students would create Analogies themselves, it seems that purposefully and consciously researching the order of the presentation of the problems would provide insight into productive learning opportunities for the students.

Although the CMIAS represent generative work of models that were initially developed in task-based interviews (Wessman-Enzinger & Mooney, 2014) and then modified in this dissertation study, it is possible that other CMIASs could still be generated in future studies. Particularly given the lack of use of Relativity, perhaps there is use of the integers or thinking about the integers that has not be demonstrated or mathematically thought about yet. Generating robust models of thinking takes time and development and more work, with other researchers perspectives of the CMIAS, is needed.

From this study, it seems that Bookkeeping and Counterbalance are connected to magnitude-based reasoning, although order-based thinking is a necessary component as well. Similarly, it seems that Translation and Relativity are connected to order-based reasoning. Research is needed to establish a tighter connection of the influence of order and magnitude on the use and development of the CMIAS.

An important result from the initial descriptions of CMIAS was the various ways to conceptualize and thinking about zero. For example, with Bookkeeping, zero represents an action or a status of neither a gain nor loss. With Counterbalance, zero represents a state of neutralization. With Translation, zero represents either a null movement or a location. With Relativity, zero presents an unknown reference. During the Group Sessions of this study, there were many unprompted occurrences of the students wondering, debating, and using zero in unexpected ways. For example, in one of the Group Sessions the students intentionally left zero off of their thermometer.

After the data in this dissertation study was collected, Wessman-Enzinger & Tobias (2015) extended the problem types of Marthe (1979) for Translation and some Relativity in the context of temperature in a different study. Both the challenges of this dissertation study with the individual context sessions and the results of Marthe (1979) and Wessman-Enzinger & Tobias (2015) highlight that there needs to be more work into semantics and problem types for integers in specific contexts and broadly in order to design tasks better in the future. If we wish to understand the CMIAS that students use when solving contextual problems, we need to know more about the contextual problem types. Although there is substantial literature on addition and subtraction types, these problem types are all focused on the addition and subtraction of positive numbers, which are significantly different contextual problems since they do not have to incorporate the ideas of opposites. The design of this study could have been substantially better if more was known and utilized about different contextual problem types.

The CMIAS reported on in this dissertation are for integer addition and subtraction. As a field we need to learn more about how students think about integer multiplication and division. Although the CMIAS would certainly be related to the to Conceptual Models for Integer Multiplication and Division (CMIMD), there would be need to be modifications to some of the existing conceptual models and new conceptual models to appropriately describe thinking about integers multiplicatively. For example, although Translation highlights that integers can be treated as a vector in integer addition and subtraction, integers are treated as either vectors or scalars multiplicatively. This highlights that further work into the development the CMIMD is needed.

Final Remarks

I designed this dissertation study to explore thinking about operations with integers, and specifically the CMIAS, something I hope to pursue as my lifetime research agenda. Through this dissertation study the CMIAS were modified to include: Bookkeeping, Counterbalance, Translation, Relativity, Proceduralization, Analogy, and Algebraic Reasoning. These CMIAS describe students' mathematical thinking and uses of integer addition and subtraction. Some of these CMIAS are used more when student generate contexts and other are utilized more as a students solve open number sentences. Students generated unique and sophisticated ways to add and subtract integers. Students need these opportunities to learn and develop conceptions in school mathematics with facilitation of their thinking and patience with significant time. Although it is the cultural mentality within the US that learning about the integers entails learning to operate correctly and learning to operate correctly is certainly important, learning about the integers is so much more than simply correct or incorrect responses. Learning about the integers also entails using different CMIAS and acquiring flexibility in that use. The CMIAS are a tool for researchers to describe student thinking, think about student thinking about integers, and formulate future research; however, the CMIAS have potential to benefit teachers and teacher educators as well. For teachers and teacher educators, the CMIAS could represent a tool for understanding what learning about integer addition and subtraction means. We can not provide opportunities for authentic learning where students are facilitated in their invented strategies, if as a research community we are not describing what it means to understand integer addition and subtraction. I believe the CMIAS are just as important to learning as the operations themselves.

REFERENCES

Altiparmak, K., & Özdoğan, E. (2010). A study on the teaching of the concept of negative numbers. *International Journal of Mathematical Education in Science and Technology*, *41*(1), 31–47.

Ayres, P. (2000). An analysis of bracket expansion errors. In T. Nakahara, & M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 25–32). Hiroshima, Japan: Nishiki Print Co., Ltd.

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal, 93*(4), 373–397.

Barrett, J. E., Sarama, J., Clements, D. H., Cullen, C., McCool, J., Witkowski-Rumsey, C., & Klanderman, D. (2012). Evaluating and improving a learning trajectory for linear measurement in elementary grades 2 and 3: A longitudinal study. *Mathematical Thinking and Learning*, *14*(1), 28–54.

Battista, M. T. (1983). A complete model for operations on integers. *Arithmetic Teacher, 30*(9), 26–31.

Bazzini, L. (1990). Examples of incorrect use of analogy in word problems. In G. Booker & P. Cobb (Eds.), *International Group for the Psychology of Mathematics Education Proceedings Fourteenth PME Conference* (Vol. 3, pp. 175–181). Mexico: Program Committee of the 14th PME Conference.

Beatty, R. (2010). Behind and below zero: Sixth grade students use linear graphs to explore negative numbers. In P. Brosnan, D. B. Erchick, & L. Flevares (Eds.), *Proceedings of the 32nd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 219–226). Columbus, OH: The Ohio State University.

Bell, A. (1982). Teaching theories in mathematics. In A. Vermandel (Ed.), *Proceedings of the 6th Conference of the International Group for the Psychology of Mathematics Education* (pp. 207–213). Antwerp, Belgium: PME.

Bell, A. (1984). Short and long term learning experiments in diagnostic teaching design. In B. Southwell (Ed.), *Proceedings of the 8th Conference of the International*

Conference for the Psychology of Mathematics Education (pp. 55–62). Sydney, Australia: International Group for the Psychology of Mathematics Education.

Bell, A. (1993). Principles for the design of teaching. *Educational Studies in Mathematics*, *24*(1), 5–34.

Bell, A., O'Brien, D., & Shiu, C. (1980). Designing teaching in light of research of understanding. In R. Karplus (Ed.), *Proceedings of the 4th Conference of the International Conference for the Psychology of Mathematics Education* (pp. 119–125). Berkeley, California: University of California.

Bishop, J. P., Lamb, L L. C., Philipp, R. A., Schappelle, B. P., & Whitacre, I. (2010). A developing framework for children's reasoning about integers. In P. Brosnan, D. Erchick, & L. Flevares (Eds.) *Proceedings of the 32nd Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. VI, pp. 695–702). Columbus, OH: The Ohio State University.

Bishop, J. P., Lamb, L. L. C., Philipp, R. A., Schappelle, B. P., & Whitacre, I. (2011). First graders outwit a famous mathematician. *Teaching Children Mathematics, 17*(6), 350–358.

Bishop, J. P., Lamb, L. L., Philipp, R. A., Whitacre, I., Schappelle, B. P., & Lewis, M. L. (2014a). Obstacles and affordances for integer reasoning: An analysis of children's thinking and the history of mathematics. *Journal for Research in Mathematics Education*, *45*(1), 19–61.

Bishop, J. P., Lamb, L. L., Philipp, R. A., Whitacre, I., & Schappelle, B. P. (2014b). Using order to reason about negative numbers: the case of Violet. *Educational Studies in Mathematics*, *86*(1), 39–59.

Bofferding, L. (2010). Addition and subtraction with negatives: Acknowledging the multiple meanings of the minus sign. In P. Brosnan, D. Erchick, & L. Flevares (Eds.), *Proceedings of the 32nd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 703–710). Columbus, OH.

Bofferding, L. (2011). Challenging fact family reasoning with instruction in negative numbers. In L. Wiest & T. Lamberg (Eds.), *Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1109–1116). Reno, NV: University of Nevada.

Bofferding, L. (2012). Transitioning from whole numbers to integers. In L. Van Zoest, J.

Lo, R. & J. Kratzy (Eds.), *Proceedings of the 34th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 935–942). Kalamazoo, MI: Western Michigan University.

Bofferding, L. (2014). Negative integer understanding: Characterizing first graders' mental models. *Journal for Research in Mathematics Education, 45*(2), 194–245.

Bofferding, L., & Alexander, A. (2011, April). *Nothing is something: First graders' use of zero in relation to negative numbers*. Paper presented at the annual meeting of the American Educational Research Association, New Orleans, LA.

Bofferding, L., & Hoffman, A. (2014). Learning negative integer concepts: Benefits of playing linear board games. In S. Oesterle, C. Nicol, P. Liljedahl, & D. Allan (Eds.), *Proceedings of the joint meeting of PME 38 and PME-NA 36* (Vol. 2, pp. 169–176). Vancouver, Canada: PME.

Bofferding, L., & Richardson, S. E. (2013).Investigating integer addition and subtraction: A task analysis. In M. Martinez & A. Superfine (Eds.), *Proceedings of the 35th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 111–118). Chicago, IL: University of Illinois at Chicago.

Bofferding, L., & Wessman-Enzinger, N. M. (2015, April). Solutions to the integers: Draw or discard game. *Teaching Children Mathematics*, *21*(8), 460–463.

Bofferding, L., Wessman-Enzinger, N. M., Gallardo, A., Salinas, G., & Peled, I. (2014). Negative numbers: Bridging contexts and symbols. In S. Oesterle, C. Nichol, P. Liljedahl, & D. Allan, *Proceedings of the joint meeting of PME 38 and PME-NA 36* (Vol. 1, p. 204). Vancouver, Canada: PME.

Bolyard, J., & Moyer-Packenham, P. (2006). The impact of virtual manipulatives on student achievement in integer addition and subtraction. In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez (Eds.), *Proceedings of the 28th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 879–881). Mérida, Yucatán, México: Universidad Pedagógica Nacional.

Booth, L. R. (1989). Grade 8 students' understanding of structural properties in mathematics. In G. Vergnaud, J. Rogalski, & M. Artique (Eds.), *Actes de la 13eme conference internationale* (Vol. 1, pp. 141–148). Paris, France: Psychology of Mathematics Education.

Brasiel, S. J. (2011). The relationship between teacher pedagogical content knowledge and student understanding of integer operations.. In L. R. Wiest & T. Lamberg (Eds.), *Proceedings of the 33rd Annual Meeting of the North American Chapter of* *the International Group for the Psychology of Mathematics Education* (pp.1050–1058). Reno, NV: University of Nevada, Reno.

Brodie, K. (2007). Teaching with conversations: Beginnings and endings. *For the Learning of Mathematics, 27*(1), 17–23.

Bruno, A., & Martinon, A. (1996). Beginning learning negative numbers. In L Puig. & A. Gutierrez (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 161–168). Valencia, Spain: PME.

Buswell, G. T., Brownell, W. A., Lenore, J. (1938). *Living arithmetic, grade eight*. Boston, MA: Ginn and Company.

Carr, K., & Katterns, B. (1984). Does the number line help? *Mathematics in School*, 30–34.

Carraher, D., Schliemann, A. D., & Brizuela, B. M. (2001). Can young students operate on unknowns? In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 130–138). Utrecht, The Netherlands: PME.

Case, R. (1996). Introduction: Reconceptualizing the nature of children's conceptual structures and their development in middle childhood. *Monographs of the Society for Research in Child Development*, *61*(1–2), 1–26.

Chiu, M. M. (2001). Using metaphors to understand and solve arithmetic problems: Novices and experts working with negative numbers. *Mathematical Thinking and Learning*, *3*(2–3), 93–124.

Chrysostomou, M., & Mousoulides, N. (2010). Pre-service teachers' knowledge of negative numbers. In M. Pinto & T. Kawakaki (Eds.), *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 265–272). Belo Horizonte, Brazil: PME.

Davidson, P. (1987). How should non-positive integers be introduced in elementary mathematics? In N. Herscovics & C. Kieran (Eds.), *Proceedings of the 11th Annual Meeting for the International Group for Pyschology of Mathematics Education* (pp. 430–436). Montreal, Canada: University of Quebec, Montreal.

Day, J., & Thomson, J. B. (1843). *Elements of algebra, being an abridgment of Day's algebra adapted to the capacities of the youth, and the method of instruction, in schools and academies.* New Haven, CT: Durrie & Peck.

Dehaene, S. (1997). *The number sense: How the mind creates mathematics*. New York, NY: Oxford University Press.

Dienes, Z. P. (1964). *The power of mathematics*. London, England: Hutchinson Educational.

Dienes, Z. P. (2000). The theory of the 6 stages of learning with integers. *Mathematics in School, 29*(2), 27–32.

Durell, F., & Robbins, E. R. (1897). *A school algebra*. New York, NY: Charles E. Merrill Co.

Ekol, G. (2010). Operations with negative integers in a dynamic geometry environment. In M. Pinto & T. Kawakaki (Eds.), *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 337–344). Belo Horizonte, Brazil: PME.

Ernest, P. (1985). The number line as a teaching aid. *Educational Studies in Mathematics*, *16*(4), 411–424.

Featherstone, H. (2000). "-Pat + Pat = 0": Intellectual play in elementary mathematics. *For the Learning of Mathematics, 20*(2), 14–23.

Fischbein, E. (1987). *Intuition in science and mathematics*. Dordrecht, The Netherlands: D. Reidel Publishing Company.

Frand, J. L., & Granville, E. B. (1978). *Theory and applications of mathematics for teachers*. Belmont, CA: Wadsworth Publishing Company.

Gallardo, A. (1994). Negative numbers in algebra. In J. de Ponte & J. Matos (Eds.), *Proceedings for the 18th International Conference for the Psychology of Mathematics Education* (pp. 376–383). Lisbon, Portugal: PME.

Gallardo, A. (1995). Negative numbers in the teaching of arithmetic. Repercussions in elementary algebra. In D.T. Owens, M. K. Reed, & G. M. Millsaps (Eds.), *Proceedings of the 17th Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 159–165). Columbus, OH: PME-NA.

Gallardo, A. (2002). The extension of the natural-number domain to the integers in the transitions from arithmetic to algebra. *Educational Studies in Mathematics*, *49,* 171–192.

Gallardo, A. (2003). "It is possible to die before being born." Negative integers subtraction: A case study. In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the Joint Meeting of PME 27 and PME-NA 25* (Vol. 2, pp. 405–411). Honolulu, HI.

Gallardo, A. (2008). Historical epistemological analysis in mathematical education: Negative numbers and the nothingess. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepúlveda (Eds.), *Proceedings of the Joint Meeting of PME 32 and PME-NA 30* (Vol. 1, pp.106–118). Morelia, Michoacán, México: Cinvestav-UMSNH.

Gallardo, A., & Hernandez, A. (2005). The duality of zero in the transtion from arithmetic to algebra. In H. Chick, & J. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 17–24). Melbourne, Australia: PME.

Gallardo, A., & Rojano, T. (1987). Common difficulties in the learning of algebra among children displaying low and medium pre-algebraic proficiency levels. In N. Herscovics & C. Kieran (Eds.), *Proceedings of the 11th Annual Meeting for the International Group for Pyschology of Mathematics Education* (pp. 301–307). Montreal, Canada: University of Quebec, Montreal.

Gallardo, A. & Rojano, T. (1994). School algebra, syntactic difficulties in the operativity with negative numbers. In D. Kirshner (Ed.), *Proceedings of the 16th Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 159–165). Baton Rouge, LA: PME-NA.

Gee, J. P. (2011). *An introduction to discourse analysis: Theory and method* (3rd ed.). New York, NY: Routledge.

Grady, M. B. (1978). A manipulative aid for adding and subtracting integers. *Arithmetic Teacher*, *26*(3), 40.

Guerrero, A., & Martinez, E. D. (1982). Additive and subtractive aspects of the comparison relationship. In A. Vermandel (Ed.), *Proceedings of the 6th Conference of the International Group for the Psychology of Mathematics Education* (pp. 150–155). Antwerp, Belgium: PME.

Hativa, N., & Cohen, D. (1995). Self learning of negative number concepts by lower division elementary students through solving computer-provided numerical problems. *Educational Studies in Mathematics, 28*(4), 401–431.

Harkin, J. B., & Rising, G. R. (1973). Some psychological and pedagogical aspects of mathematical symbolism. *Educational Studies in Mathematics*, *5*(1), 255–260.

Heeffer, A. (2011). Historical objections against the number line. *Science & Education, 20*(9), 863–880.

Hefendehl-Hebeker, L. (1991). Negative numbers: Obstacles in their evolution from intuitive to intellectual constructs. *For the Learning of Mathematics, 11*(1), 26–32.

Henley, A. T. (1999). *The history of negative numbers.* Unpublished doctoral dissertation. South Bank University.

Herbst, P. (1997). The number-line metaphor in the discourse of a textbook series. *For the Learning of Mathematics*, 17(3), 36–45.

Hitchcock, G. (1997). Teaching the negatives, 1870–1970: A medley of models. *For the Learning of Mathematics*, *17*(1), 17–42.

Human, P., & Murray, H. (1987). Non-concrete approaches to integer arithmetic. In N. Herscovics & C. Kieran (Eds.), *Proceedings of the 11th Annual Meeting for the International Group for Psychology of Mathematics Education* (pp. 437–443). Montreal, Canada: University of Quebec, Montreal.

Iannone, P., & Cockburn, A. D. (2006). Fostering conceptual mathematical thinking in the early years: A case study. In J. Novotna, H. Moraova, M. Kratka, & N. Stehlikova (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 329–336). Prague, Czech Republic: PME.

Jablonka, E., Wagner, D., & Walshaw, M. (2013). Theories for studying social, political and cultural dimensions of mathematics education. In M. A. Clements, A. Bishop, C. Keitel-Kreidt, J. Kilpatrick, & F. Leung (Eds.), *Third International Handbook of Mathematics Education* (pp. 41–67). New York, NY: Springer.

Janvier, C. (1985). Comparison of models aimed at teaching. In L. Streefland (Eds.), *Proceedings of the 9th Conference of the Psychology of Mathematics Education* (pp. 135–140). Noordwijkerhout, The Netherlands: International Group for the Psychology of Mathematics Education.

Kilhamn, C. (2009). Making sense of negative numbers through metaphorical reasoning. In C. Bergsten, B. Grevholm, & T. Lingefjärd (Eds.), *Perspectives on mathematical knowledge. Proceedings of madif6* (pp. 30–35). Linköping, Sweden: SMDF.

Kilhamn, C. (2011). *Making sense of negative numbers.* Göteborg, Sweden: Acta Unversittatis Gothenburgensis.

Kilpatrick, S., & Swafford, J. Findell, B. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Koukkoufis, A. & Williams, J. (2006). Integer instruction: A semiotic analysis of the "Compensation strategy." In Novotna, J, Moraova, H., Kratka, M. & Stehlikova, N. (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 473–480). Prague: PME.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York, NY: Basic books.

Lamb, L. C., Bishop, J. P., Philipp, R. A., Schappelle, B. P., Whitacre, I., & Lewis, M. (2012). Developing symbol sense for the minus sign. *Mathematics Teaching in the Middle School*, *18*(1), 5–9.

Lamb, L. C. & Thanheiser, E. (2006). Understanding integers: Using balloons and weights software. In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez (Eds.), *Proceedings of the 28th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, p. 163). Mérida, Yucatán, México:Universidad Pedagógica Nacional.

Lamb, L., Bishop, J., Philipp, R., Whitacre, I., Stephan, M., Bofferding, L., Lewis, J., Brickwedde, J., Bagley, S., & Schappelle, B. (2013). Building on the emerging knowledge base for teaching and learning in relation to integers. In M. Martinez & A. Castro Superfine (Eds.), *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1362–1366). Chicago, IL: University of Illinois at Chicago.

Langrall, C. (2013). Identity, power, and stewardship: Perspectives of a new editor. *Journal for Research in Mathematics Education*, *44*(1), 3–4.

Larsen, S. & Saldanha, L. (2006). Function composition as combining transformations: Lessons learned from the first iteration of an instructional experiment. In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez (Eds.), *Proceedings of the 28th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp.790–797). Mérida, Yucatán, México: Universidad Pedagógica Nacional.

Leatham, K. R. (2015). Observations on citation practices in mathematics education research. *Journal for Research in Mathematics Education*, *46*(3), 253–269.

Liebeck, P. (1990). Scores and forfeits. *Educational Studies in Mathematics, 21*(3), 221–239.

Linchevski, L. & Williams, J. (1996). Situated intuitions, concrete manipulations and the construction of mathematical concepts: The case of integers. *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 265–272). Valencia, Spain: PME.

Linchevski, L., & Williams, J. (1999). Using intuition from everyday life in "filling" in gaps in children's extension of their number concept to include negative numbers. *Educational Studies in Mathematics, 39*, 131–147.

Loomis, E. (1857). *Treatise on algebra*. (12th ed.) New York, NY: Harper & Brothers.

Marthe, P. (1979). Additive problems and directed numbers. In D. Tall (Ed.), *Proceedings of the 3rd Conference of the International Group for the Psychology of Mathematics Education* (pp. 317–323). Coventry, England: PME.

Marthe, P. (1982). Research on the appropriation of the additive group of directed numbers. In A. Vermandel (Ed.), *Proceedings of the 6th Conference of the International Group for the Psychology of Mathematics Education* (pp. 162–167). Antwerp, Belgium: PME.

Merriam, S. B. (1998). *Qualitative research and case study applications in education*. San Francisco, CA: Jossy-Bass.

Mukhopadhyay, S. (1997). Storytelling as sense-making: Children's ideas about negative numbers. *Hiroshima Journal of Mathematics Education*, *5*, 35–50.

Mukhopadhyay, S., Resnick, L. B., & Schauble, L. (1990). Social sense-making in mathematics; Children's ideas of negative numbers. In G. Booker & P. Cobb (Eds.), *International Group for the Psychology of Mathematics Education Proceedings of the Fourteenth PME Conference* (Vol. 3, pp. 281–288). Oaxtepex, Mexico: Program Committee of the 14th PME Conference.

Murray, J. C. (1985). Children's informal conceptions of integers. In L. Streefland (Eds.), *Proceedings of the 9th Conference of the Psychology of Mathematics Education* (pp. 147–153). Noordwijkerhout, The Netherlands: International Group for the Psychology of Mathematics Education.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Governors Association Center for Best Practices. Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Washington DC: Author. Retrieved from http://www.corestandards.org/the-standards.

Nicodemus, R. (1993). Transformations. *For the Learning of Mathematics, 13*(1), 24–29.

Nurnberger-Haag, J. (2007). Integers made easy: Just walk it off. *Mathematics Teaching in the Middle School*, *13*(2), 118–121.

Orlov, K. (1971). Experimental verification of the use of the mathematical balance in secondary teaching. *Educational Studies in Mathematics*, *3*(2), 192–205.

Peled, I. (1991). Levels of knowledge about negative numbers: Effects of age and ability. In F. Furinghetti (Ed.), *Proceedings of the 15th International Conference for the Psychology in Mathematics Education* (Vol. 3, pp. 145–152). Assisi, Italy: Conference Committee.

Peled, I., & Carraher, D. W. (2008). Signed numbers and algebraic thinking. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 303–328). New York, NY: Routledge.

Peled, I., Mukhopadhyay, S., & Resnick, L. (1989). Formal and informal sources of mental models for negative numbers. In G. Vergnaud, J. Rogalski, & M. Artique (Eds.), *Actes de la 13eme conference internationale* (Vol. 3, pp. 106–110). Paris, France: Psychology of Mathematics Education.

Piaget, J. (1948). *To understand is to invent: The future of education*. New York, NY: Viking Press.

Piaget, J. (1952). *The child's conception of number*. London, UK: Routledge & Kegan Paul, Ltd.

Poirier, L. & Bednarz, N. (1991). Mental models and problem solving: An illustration with complex arithmetical problems. In R. Underhill (Ed.), *Proceedings of the 13th Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 133–139). Blacksburg, VA: Christiansburg Printing Company.

Reeves, C. A., & Webb, D. (2004). Balloons on the rise: A problem-solving introduction to integers. *Mathematics Teaching in the Middle School*, *9*(9), 476–482.

Roswell, D. W., & Norwood, K. S. (1999). Student-generated multiplication word problems. In O. Zaslavsky (Ed.), *Proceedings of the 23rd International Conference for the Psychology in Mathematics Education* (Vol. 4, pp. 121–128). Haifa, Israel: Conference Committee.

Ryan, J. T., Williams, J. S., & Doig, B. A. (1998). National tests: Educating teachers about their children's mathematical thinking. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 81–88). Stellenbosch, South Africa: University of Stellenbosch.

Sasaki, T. (1993). The constructing meanings by social interaction in mathematical teaching. In I. Hirabayashi (Ed.), *Proceedings of the 17th International Conference on the Psychology of Mathematics Education* (Vol. 2, pp. 262–269). Tsukuba, Japan: University of Tsukuba.

Saxe, G. B., Diakow, R., & Gearhart, M. (2013). Towards curricular coherence in integers and fractions: A study of the efficacy of a lesson sequence that uses the number line as the principle representational context. *ZDM Mathematics Education, 45*(3), 343–364.

Schorr, R. Y., & Alston, A. S. (1999). Keep change change. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 169–176). Haifa, Israel: Israel Institute of Technology.

Schubring, G. (2005). *Conflicts between generalization, rigor, and intuition: Number concepts underlying the development of analysis in 17–19th century France and Germany*. New York, NY: Springer.

Seidelmann, A. L. (2004). Students' conceptions of zero (Unpublished doctoral dissertation). Illinois State University, Normal, IL.

Selter, C., Prediger, S., Nührenbörger, M., & Hußmann, S. (2012). Taking away and determining the difference—A longitudinal perspective on two models of subtraction and the inverse relation to addition. *Educational Studies in Mathematics*, *79*(3), 389–408.

Sfard, A., & Avigail, S. (2006). When the rules of discourse change, but nobody tells you – The case of a class learning about negative numbers. Mimeograph. Retrieved from http://eprints.ioe.ac.uk/4310/1/negatives_-_22_May_06.pdf.
–

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge, MA: Cambridge University Press.

Sfard, A. (Personal communication, February 27, 2014).

Shore, F. S. (2005). Operating with integers: A familiar model under new contexts. *Ohio Journal of School Mathematics*, 52, 7–11.

Smiddy, J. (2008). *Everyday mathematics and its use as curricular reform to stimulate stronger school performance*. (Doctoral dissertation, Pacific Lutheran University).

Smith, L. B., Sera, M., & Gattuso, B. (1988). The development of thinking. In R. Sternberg & E. Smith (Eds.), *The psychology of human thought* (pp. 366–391). Cambridge, UK: Cambridge University Press.

Steffe, L., & Olive, J. (2009). *Children's fractional knowledge*. Springer Science & Business Media.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education,* (pp. 267–306). Mahwah, NJ: Lawrence Erlbaum Associates.

Stephan, M., & Akyuz, D. (2012). A proposed instructional theory for integer addition and subtraction. *Journal for Research in Mathematics Education, 43*(4), 428–464.

The NRICH Project. (2012). *The history of negative numbers*. Retrieved from http://nrich.maths.org/5961

Thomaidis, Y., & Tzanakis, C. (2007). The notion of historical “parallelism” revisited: Historical evolution and students’ conception of the order relation on the number line. *Educational Studies in Mathematics*, *66*(2), 165–183.

Thompson, P. W. (2013). Advances in research on quantitative reasoning. In R. Mayes & L. Hatfield (Eds.), *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context* (pp. 143–148). Laramie, WY: University of Wyoming.

Thompson, P. W., & Dreyfus, T. (1988). Integers as transformations. *Journal for Research in Mathematics Education*, *19*(2), 115–133.

Tillema, E. S. (2012). What is the difference? Using Contextualized Problems. *Mathematics Teaching in the Middle School*, *17*(8), 472–478.

Ulrich, C. L. (2012). *Additive relationships and signed quantities* (Doctoral dissertation, University of Georgia).

Ulrich, C. L. (2013). The addition and subtraction of signed quantities. In R. Mayes & L. Hatfield (Eds.), *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context* (pp. 127–141). Laramie, WY: University of Wyoming.

Varma, S., & Schwartz, D. L. (2011). The mental representation of integers: An abstract-to-concrete shift in the understanding of mathematical concepts. *Cognition*, *121*(3), 363–385.

Vergnaud, G. (1982). Cognitive psychology and didactics: Signified/signifier and problems of reference. In A. Vermandel (Ed.), *Proceedings of the 6th Conference of the International Group for the Psychology of Mathematics Education* (pp. 70–76). Antwerp, Belgium: PME.

Vig, R., Murray, E., & Star, J. R. (2014). Model breaking points conceptualized. *Educational Psychology Review*, *26*(1), 73–90.

Vlassis, J. (2001). Solving equations with negatives or corssing the formalizing gap. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 374–382). Utrecht, The Netherlands: PME.

Vlassis, J. (2008). The role of mathematical symbols in the development of number conceptualization: The case of the minus sign. *Philosophical Psychology*, *21*(4), 555–570.

Wessman-Enzinger, N. M. (2013). Contexts of student constructed stories about negative integers. In M. Martinez & A. Castro Superfine (Eds.), *Proceedings of the 35th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (p. 167). Chicago, IL: University of Illinois at Chicago.

Wessman-Enzinger, N. M. (2015). *Evolution of the number line in North American school mathematics*. Manuscript in preparation.

Wessman-Enzinger, N. M. (in press). Alice's drawings for integer addition & subtraction open number sentences. *Proceedings of the 37th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

Wessman-Enzinger, N. M., & Bofferding, L. (2014). Integers: Draw or discard! game. *Teaching Children Mathematics, 20*(8), 476–480.

Wessman-Enzinger, N. M., & Mooney, E. S. (2014). Making sense of integers through story-telling. *Mathematics Teaching in the Middle School, 20*(4), 203–205.

Wessman-Enzinger, N. M., & Mooney, E. S. (2015). *Conceptual models of integers*. Submitted for publication.

Wessman-Enzinger, N. M., & Tobias, J. (2015). Preservice teachers' temperature stories for integer addition and subtraction. In K. Beswick, T. Muir, & J. Wells (Eds.), *Proceedings of the 39th Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 289–296). Hobart, Australia: PME.

Wheeler, M. M., & Feghali, I. (1983). Much ado about nothing: Preservice elementary school teachers' concept of zero. *Journal for Research in Mathematics Education*, *14*(3), 147–155.

Wheeler, D., Nesher, P., Bell, A., & Gattegno, C. (1981). A research programme for mathematics education (I). *For the Learning of Mathematics, 2*(1), 27–29.

Whitacre, I., Bishop, J. P., Lamb, L. L. C., Philipp, R. A., Schappelle, B. P., & Lewis, M. (2011). Integers: History, textbook approaches, and children's productive mathematical intuitions. In L. Weist & T. Lamberg (Eds.), *Proceeding of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 913–920). Reno, NV: University of Nevada.

Whitacre, I., Bishop, J. P., Lamb, L. L. C., Philipp, R. A., Schappelle, B. P., & Lewis, M. (2012a). Happy and sad thoughts: An exploration of children's integer reasoning. *The Journal of Mathematical Behavior, 31*, 356–365.

Whitacre, I., Bishop, J. P., Lamb, L. L. C., Philipp, R. A., Schappelle, B. P., & Lewis, M. (2012b). What sense do children make of negative dollars? In L. Van Zoest, J. Lo., & J. Kratky (Eds.), *Proceedings of the 34th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 958–964). Kalamazoo, MI: Western Michigan University.

Whitacre, I., Pierson Bishop, J., Lamb, L. L., Philipp, R. A., Bagley, S., & Schappelle, B. P. (2014). Negative of my money, positive of her money: Secondary students' ways of relating equations to a debt context. *International Journal of Mathematical Education in Science and Technology*, 1–16.

Woo, J. H. (2007). School mathematics and cultivation of mind. In J. Woo, H. Lew, K. Park, & D. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 65–93). Seoul, Korea: PME.

APPENDIX A

PILOT STUDY ITEMS

Teacher Instructions:

Thank you so much for agreeing to participate! ☺ Here are some important points to keep in mind for the study:

1. Make sure the students do not put their name on the paper. If the students accidentally put their name on the paper, please use a Sharpie or otherwise to black out their names.

2. Before the students work on their worksheets, the only instruction about the activity should be the following "whole-class example." Also, before the students start, encourage them that all that is important is *how they think* about the number sentences and that there isn't a "right" answer.

Whole-Class Example

Consider the number sentence:

$$6 + 4 = 10$$

A possible story is:

> I had 6 books before my birthday. At my birthday party I received 4 books. Then, I had ten books.

3. When the students are working on the worksheet, they may ask or say things like, "What should I do about the minus sign?" or "I'm not sure what minus and minus means." Please encourage the students to do their best to share what they think it means and that there isn't a "right answer."

4. If they can't make sense of the problem, then the students can write "no story."

5. After the worksheets are collected, you may like to make copies for your own records or you may want to use the results for classroom discussion.

Please provide a story for the number sentences. If you can't think of a story that would work, please write the phrase, "No story."

1. $16 - 4 = 12$

2. $-17 + 12 = -5$

3. 18 + -13 = 5

4. 8 – 20 = -12

5. -2 – 3 = -5

6. -14 – -20 = 6

7. $-6 + -9 = -15$

8. $23 + 5 = 28$

APPENDIX B

WRITTEN PRE-ASSESSMENT 1

Name ___________________________________ Date _______________________

Consider the number sentences below. For each number sentence, write a story that you think matches the number sentence and then solve the number sentence. If you can't write a story for the number sentence write "No Story." If you can't solve the number sentence write "No."

1. $30 - 15 = \square$

Story:

Solve:

2. $-20 + 17 = \boxed{}$

Story:

Solve:

3. $3 + -12 = \boxed{}$

Story:

Solve:

4. $-9 + -15 = \square$

Story:

Solve:

5. $-4 + 16 = \square$

Story:

Solve:

6. $-1 - 5 = \square$

Story:

Solve:

7. $7 - 15 = \square$

Story:

Solve:

8. $-2 - -1 = \square$

Story:

Solve:

9. $-3 - -8 = \square$

Story:

Solve:

10. $23 + 26 = \square$

Story:

Solve:

APPENDIX C

WRITTEN PRE-ASSESSMENT 2

Name ___________________________________ Date _______________________

Directions: *For each of the following problems, write a number sentence that you think matches the situation. Provide a solution to the problem and include any work or pictures that helped you think about the problem.*

1. A scuba diver was swimming 200 feet below sea level and a sea gull was flying 30 feet above the sea. How far apart were the scuba diver and the sea gull?

Number Sentence: ____

Solution: ____

2. Sammy borrowed $15 from her sister. Sammy paid $8 of it back to her sister. How much money does Sammy owe her sister?

Number Sentence: ____

Solution: ____

3. Sebastian has $300 in credit card debt and $100 dollars in his saving account at the bank. What is Sebastian's net worth?

Number Sentence: ____

Solution: ____

4. This week, Liam did 8 bad deeds and 15 good deeds at school. What kind of week did Liam have?

Number Sentence: ____

Solution: ____

5. Angela and her dad decided to plant a tree in their backyard. Angela found a spot in the yard that she wanted to plant the tree. Her dad told her to move the tree from that spot to the right 10 feet. Angela moved the tree to the position that her dad wanted, but then she decided to move it 12 feet to the left. Where did Angela eventually end up planting the tree?

Number Sentence: ____

Solution: ____

6. There are 3 electrons and 3 protons in an atom. What is the charge of the atom?

Number Sentence: ____

Solution: ____

7. The Red Birds and Dragons were playing against each other in a baseball game. The Red Birds were down 4 runs in the first inning and gained 6 runs against the Dragons in the second inning. How did the Red Birds start the third inning?

Number Sentence: ____

Solution: ____

8. It was 8° Fahrenheit outside. The temperature dropped 20° Fahrenheit. What is the temperature now?

Number Sentence: ____

Solution: ____

APPENDIX D

CONTENT AND DEVELOPMENT OF GROUP SESSIONS

Recognizing that Bookkeeping was the most utilized CMIAS in the pilot study, Group Session 1 in this study began with contexts that were intended to promote Bookkeeping. The development of the content in each of the Group Sessions was contingent on what happened in the prior Group Sessions. Although the Group Sessions and protocols used during those Group Sessions were developed in process and dependent upon the students' contributions, the design of the protocol did use contexts that were tied to the conceptual models in Table 12. Table 12 provides an outline for the CMIAS that were promoted during the Group Sessions by contexts. Four conceptual models were introduced to the students; however, it is recognized that the students may have used other models of thinking about integers, despite the promotion of these conceptual models with contexts. It is the hope of this study to use Group Sessions as exposure to CMIAS and the ways that other students make sense of the integers during these sessions.

The structure of the Group Sessions, the types of problems types addressed, and even what conceptual models were promoted during the sessions was developed during the 12 weeks of the teaching experiment. The Group Sessions were designed to support the students' learning of integer addition and subtraction, while also promoting four conceptual models through contextual situations. The witness and myself, the teacher-researcher, collaborated over the design of each of the nine Group sessions. The development and collaborations of each of these Group sessions is discussed next.

Group Session 1

The purpose of this Group Session was to present two contexts that promoted Bookkeeping. This Group Session began with contexts that promoted Bookkeeping because the students in the pilot study utilized this conceptual model the most. Two contexts were chosen (i.e., the “wanting” baseball cards and the “losing” baseball cards) to promote thinking broadly about gains and losses of something. The purpose for asking the students to think about Pete and Allyson’s “thinking” in this Group Session was to facilitate the students’ Bookkeeping thinking. It was expected that the students in this study would be able to participate actively in this Group Session based on their written assessments and one of the students mentioned Bookkeeping ideas connected to a video game during the first interview. All three students demonstrated some proficient and productive strategies for adding integers in the pre-assessment interview. For this reason, the teaching experiment focused on integrating their ideas about addition with Bookkeeping.

Teacher-Researcher Directions: **Consider the following number sentence. Solve it and explain how you are thinking about it.**

$$-20 + 5 = \square$$

- Have each of the students explain their thinking.
- Ask the student if they understand each other's thinking or if they agree with the other's thinking.

Pete and Allyson both wrote stories that they thought corresponded with the same open number sentence:

$$-20 + 5 = \square$$

Pete's Story	Allyson's Story
I want 20 baseball cards and I got five baseball cards, and now I still want 15 baseball cards.	I lost 20 pencils. My mom bought me 5 pencils the next day. I still need 15 pencils.

1. What do you think about Pete and Allyson's stories?

    a. Possible probing questions:
        i. Do you agree with, Pete or Allyson? Both of them? Or, neither of them? Why?
        ii. Why do you disagree with Pete or Allyson, or neither?

    b. If students disagree with either Pete or Allyson, then ask:
        i. How can you change the story to make it work?
        ii. What is a new story that would work?

    c. If students agree with either Pete or Allyson, then ask:
        i. Can you explain what Pete was thinking?
        ii. Can you explain what Allyson was thinking?

2. Using ____________________'s thinking, solve the following number sentences.

    - Switch between students participating in the teaching experiment and Pete/Allyson. For example, "Edward, can you explain how Pete would think about "-10 + 3" and what kind of answer he would give?"

- Then, ask the other students if they agree or disagree and why? For example, "Jennifer, do you agree that Pete would think that way? Why? Or, Why not?"
- Ask the students what kind of story Pete or Allyson would make up for b – d.

  a. $-10 + 3 = \square$

  b. $10 + 3 = \square$

  c. $10 + -3 = \square$

  d. $-10 + -3 = \square$

Group Session 2

The students in this study could solve the addition problems correctly in the first Group Session; however, their stories with Bookkeeping perspectives often did not match the number sentences or their answers, and the students were content with that. One purpose of this Group Session was to draw students' attention to changing number sentences and changing stories in significant ways. A context was not utilized in this Group Session; students were asked to tell a story for 11 + 6 and modify that original story for number sentences like -11 + 6. Because most students do invent stories about joining quantities, rather than Translations or other CMIAS, for problem types like 11 + 6, the task encouraged the students to modify their stories that they wrote. Since the students were modifying stories about joining or taking away discrete objects, this task promoted Bookkeeping.

Teacher-Researcher Instructions:

- ***What is your story? How does your number sentence match your story?***
- ***How did your story change?***
- ***Why did you change it that way?***
- ***How would you change A's story to fit?***

1. Write a story for the following number sentence.

$$11 + 6 = \square$$

   a) If you changed the number sentence from $11 + 6 = \square$ to $-11 + 6 = \square$, then how would your story change?

   b) If you changed the number sentence from $11 + 6 = \square$ to $-11 + -6 = \square$, then how would your story change?

   c) If you changed the number sentence from $11 + 6 = \square$ to $11 + -6 = \square$, then how would your story change?

Group Session 3

One of the purposes of this Group Session was to introduce or try to promote Counterbalance with contexts. Two problems that promoted Counterbalance and one problem that promoted Bookkeeping were given, since the students have already spent two Group Sessions with Bookkeeping. The students in this study had not appeared to use Counterbalance yet. Other purposes of this Group Session were to facilitate the students' focus on connecting number sentences to the problems and expose students to other stories and contexts. Up to this point in the groups sessions, the students in this study were mostly concerned with only their initial numbers "matching" in their stories, rather than operations matching or the outcome of their story matching the solution to the number sentence.

So far, in the Group Sessions, at this point we did not have the opportunity to talk about subtraction. First, these students had not brought subtraction up by writing it as a number sentence. Second, we did not make it past the planned addition problems to subtraction problems. This Group Session was intended to be open to talking about subtraction, since some problems, like subtraction problems, are computationally equivalent to adding a negative number; however, the opportunity did not present itself again.

As I was designing this particular Group Session, coming up with how to ask the questions at the end of the contexts was a major challenge. I oscillated between deciding to use questions and deciding to remove the questions for the problem that used Bookkeeping. The issue is that most of the questions that draw upon Bookkeeping are leading, giving a student a specific perspective to preference. Or, the questions do not aid in the use of negative numbers. For example, consider the task, “Kyle owed his brother 18 dollars. He paid him back 12 dollars today.” I could ask, “How much money does Kyle have?” or “How much money does Kyle still owe his brother?” If ask these questions, this gives the students a certain perspective to take (see, e.g., Whitacre et al., 2014). I ended up deciding to use, “What’s Kyle’s status now?” I decided to do this because I wanted to pose the most neutral of questions as possible in this Group Session. Although I did consider leaving the questions off, I decided to leave the questions because if this caused confusion, discussion, or breaks down with the negatives, then it would be documented in the field notes and transcripts. Perhaps if the questions for Bookkeeping cause problems, this is something to highlight and investigate in the future.

Teacher-Researcher Instructions: **Ask the students to consider the following situations. Ask the students to write a number sentence that they think matches the story. After students have written a number sentence, have them explain why they think it works. Then, show the students a list of number sentences (shown in the right column) and ask them to circle which ones that they think match the situation best. Ask the students to explain why they think each number sentence does match or does not match.**

In one month, Jerry had had 18 bad days and 12 good days. What kind of month did Jerry have?	$18 + 12 = \square$ $18 - 12 = \square$ $-18 + 12 = \square$ $18 + -12 = \square$ $-18 + -12 = \square$ $-18 - 12 = \square$ $-18 - -12 = \square$
Kyle owed his brother 18 dollars. He paid him back 12 dollars today. What is Kyle's situation now?	$18 + 12 = \square$ $18 - 12 = \square$ $-18 + 12 = \square$ $18 + -12 = \square$ $-18 + -12 = \square$ $-18 - 12 = \square$ $-18 - -12 = \square$
Warren did five bad deeds in the morning and then two bad deeds in the afternoon. How was Warren behaving today?	$5 + 2 = \square$ $5 - 2 = \square$ $-5 + 2 = \square$ $5 + -2 = \square$ $-5 + -2 = \square$ $-5 - 2 = \square$ $-5 - -2 = \square$

Group Session 4

One purpose of this group session was to promote Counterbalance by playing a card game, *Integers: Draw or Discard*. We started to promote Counterbalance in the third group session; however, the student did not appear to see the "neutralization" in the quantities. To emphasize this "neutralization" with a context, I decided to use a card game that uses integer cards from -8 to 8 (Bofferding & Wessman-Enzinger, 2015; Wessman-Enzinger & Bofferding, 2014). I selected this card game because the cards are integer quantities that will remain present in their hands of cards, giving the students opportunities to experience neutralization. For example, a -2 card will be compared to a 2 card and a -3 card will be compared to a 3 card in this Group Session, it will hopefully promote the “balancing” or the “neutralization” of these quantities. If a student has a hand of 2, -2, and 7, it is worth the same in this game as a hand of 3, -3, and 7. Although promoting Counterbalance is the main purpose of this Group Session, another purpose of this Group Session was to begin discussion on integer subtraction by discarding cards. To foster both addition and subtraction discussion in this session, cards were both discarded and added to ones’ hand in this game. Discarding a card is similar to subtraction. Thus, if students discard a negative integer card they may consider the effects of subtracting a negative integer.

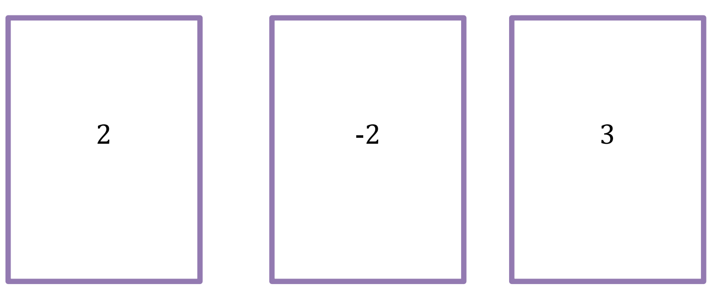
***Teacher-Researcher Instructions*: Play ten rounds of the card game, *Integers: Draw or Discard*. Use one deck of cards for three students. Provide each student with a recording sheet. After the students have played 10 rounds of the game and recorded their results, pose the following questions for discussion:**

1. What did you notice after playing the game? Did you notice anything about negative numbers?

Other Teacher-Researcher Questions:

- *Another possible probing question may be, what did you notice when you were drawing and discarding cards?*
- *Listen to the students’ dialogue during game play and possibly ask a question about something they said during game play as well.*

2. Emma had the following cards in her hand.



a. Write a number sentence for the amount of points that Emma has.

b. Emma turns over a 1 card in the center deck, do you think she should draw that card or do you think she should discard a card from her hand? Why?

1

c. Write a new number sentence for the amount of points that Emma has with her draw or discard.

3. Aaron had the following cards in his hand.

-3 -5 8

a. Write a number sentence for the points that Aaron has.

b. Aaron turns over a -2 card in the center deck, do you think that he should draw that card or do you think he should discard a card from his hand? Why?

-2

c. Write a new number sentence for the amount of points that Aaron has with his draw or discard.

Recording Sheet

What cards did you begin with? (List your starting cards here)			**Points**
Turn	**Circle What you Did**	**Number on the Card**	**Your Points so Far**
1	Add a card Discard a card		
2	Add a card Discard a card		
3	Add a card Discard a card		
4	Add a card Discard a card		
5	Add a card Discard a card		
6	Add a card Discard a card		
7	Add a card Discard a card		
8	Add a card Discard a card		
9	Add a card Discard a card		
10	Add a card Discard a card		

Group Session 5

The card game context was used again this Group Session. Not only were the students interested in this game, but this game also promoted Counterbalance by maintaining quantities (i.e., the value of the cards) in their hands. The students did not appear to be utilizing idea of neutralization in their sessions and the physical presence of the cards again was intended to help promote the counterbalance conceptual model. The game context was used in Group Session 5. The students were provided different situations that game players could encounter and were asked to respond to this. The situations that were given to the student were reminiscent of Stephan and Akyuz (2012) "net worth" problems; however, this particular problem utilized the total points rather than money. Stephan and Akyuz (2012) although utilizing money, also present a context that is Counterbalance in nature. Again, this task attempted to promote Counterbalance while discussing the addition and subtraction of integers.

1. Look at John's *Recording Sheet* from another game. He forgot to fill out all of the parts of his sheet. Help John fill in the missing parts of his recording sheet.

Recording Sheet

<table>
<tr><td colspan="3">What cards did you begin with?

(List your starting cards here)</td><td>Points

-5</td></tr>
<tr><td>Turn</td><td>Circle What you Did</td><td>Number on the Card</td><td>Your Points so Far</td></tr>
<tr><td>1</td><td>Add a card   Discard a card</td><td>-7</td><td></td></tr>
<tr><td>2</td><td>Add a card   Discard a card</td><td>3</td><td>-1</td></tr>
</table>

a. Write a number sentence for Turn 1 of John's recording sheet.

b. Write a number sentence for Turn 2 of John's recording sheet.

c. Did Turn 1 make John's amount of points bigger or smaller? Why?

d. Did Turn 2 make John's amount of points bigger or smaller? Why?

2. Look at John's *Recording Sheet* from another game. He forgot to fill out all of the parts of his sheet. Help John fill in the missing parts of his recording sheet.

Recording Sheet

What cards did you begin with? (List your starting cards here)			**Points** -5
Turn	**Circle What you Did**	**Number on the Card**	**Your Points so Far**
1	Add a card Discard a card	-7	
2	Add a card Discard a card	3	-1

a. Write a number sentence for Turn 1 of John's recording sheet.

b. Write a number sentence for Turn 2 of John's recording sheet.

c. Did Turn 1 make John's amount of points bigger or smaller? Why?

d. Did Turn 2 make John's amount of points bigger or smaller? Why?

Group Session 6

One objective of this Group Session was to try to promote Translation. The other objective of this session was to initiate thinking about the difference between Translation and Relativity. For example, some of the problems utilized an unknown referent in the problems, a distinguishing characteristic of the Relativity. Although the referent is unknown in both of the problems, which is considered Relativity, there is "movement" or Translation embedded into the context. These problems were used to see if the students applied movements or Translation to these types of contexts or struggled with the unknown referent. The questions were intentionally left out to allow for the students to have flexibility in how they want to think about the situation and allowed the students to interpret freely. Posing these tasks and omitting the question was done help distinguish between Translation and Relativity better. The context of the first problem was about moving up and down a river (Marthe, 1979). And, the context of football was used a context in the second problem in an effort to involve one of the less verbal participants in the discussion more.

Teacher-Researcher Instructions: **Ask the students to consider the following situations. Ask the students to write a number sentence that they think represents each of these situations. After discussion about the first number sentence, ask students if they can provide another number sentence. After discussion about further number sentences, provide students with a list of number sentences and ask them to circle all of the number sentences that they think match the following situation.**

1. A man paddled his canoe upstream a river. It was a very windy day. The man paddled forward for 8 miles, but the wind was too strong to continue. As he rested, he fell asleep and his canoe was blown back 12 miles.

   Number Sentences to Show Students:

   $$8 + -12 = \square$$

   $$-8 + 12 = \square$$

   $$8 - 12 = \square$$

   $$12 - 8 = \square$$

   $$-8 - -12 = \square$$

   $$-12 - -8 = \square$$

2. The Dragons were playing a football game against the Tigers. After the first play the Dragons lost 8 yards. During the second play, the Dragons were pushed back another 12 yards.

   Number Sentences to Show Students:

   $$8 + 12 = \square$$

   $$-8 + -12 = \square$$

$-8 - 12 = \square$

$8 - -12 = \square$

$12 - -8 = \square$

$-12 - 8 = \square$

Group Session 7

One of the goals of this Group Session was to promote the Translation, and possibly Relativity. This Group Session followed the third set of Individual Sessions. After the Individual Sessions, it seemed as if all three participants had misconceptions about problem types form $-a - b$ and $a - -b$, where $a, b > 0$. To address this, tasks were created that could possibly promote discussion about these problem types, in addition to trying to promote conceptual models of translation. In Group Session 6, the students brought up the number line and all three students drew pictures of number lines. Yet, hardly any of the students really used the number line in either the contextual and number sentence Individual Sessions. Since the students had spent a significant amount of time on with negative integers at this point, they started to develop their own "rules" for dealing with operating with the integers and are drawing on these rules (even in incorrect) more than other conceptual reasoning. For this reason, temperature was introduced as a context in this teaching experiment. The context of temperature has a culturally established scale and may promote the student to reason about this scale (and thus number line). The first two problems of this Group Session were designed to help students thinking about direction of adding and subtracting on a temperature scale, and thus number line. The students in this teaching experiment had been solving the problem type $-a + b$ successfully, which was used in the first problem in this Group Session;

however, the students had not been solving the problem type, -a – b successfully throughout their individual interviews. The second problem of the Group Session as intended to help promote Translation, as well as, the problem type, -a – b. The third problem in this Group Session was also designed to promote translation or relativity conceptual models. These problems included increasing and decreasing temperatures, which was thought to promote ideas of translation. The third problem differed from the first two by asking the students to consider the difference in two temperatures. Because each of the temperatures are relative, the distance between them is relative. For example, students could give a difference that is positive or negative. They may or may not attribute direction to this difference in temperature. Because of this, this problem was thought to promote Relativity. The third problem also addressed a problem type, a – -b, which was challenging for the students in their individual interviews. The problem type expected in the third problem is another challenging problem for the students. They were able to successfully subtract a negative in Group Sessions 4 and 5, promoting Counterbalance, but had not referenced that experience since. For the third problem in this Group Session, it was expected that the students would answer which place is warmer without issue and justify that positive temperatures are warmer. When asked to determine how much warmer, there were two expected strategies: (a) The students could correctly answer, by drawing a picture and writing 5 + 9; or, (b) The students could write a number sentence like, -9 – 5 and get -4, based on their past strategies. The possible probing questions for this third problem if they answered this way were:

- Besides 5 + 9, can you write another number sentence?

- Is there a number sentence that uses the negative nine that you could write? What if this problem was different?
- What if it was 5 degrees in the North Pole and 3 degrees in the South Pole?
- What kind of number sentences would you write then?
- How does that compare to what we are doing now?

This Group Session was considered to potentially be promoting ideas of both Translation and Relativity. These contextual problems may have promoted ideas of Translations because there was action of rising and dropping temperatures in the both the first and second problems. The third problem was a comparison type problem that did not involve rising or falling temperatures. All three problems utilized the Fahrenheit temperature scale, which is a relative scale. Thus, one of the main aims of this Group Session in terms CMIAS was to use problems with action (i.e., Translation) and non-action (i.e., Translation or Relativity). Also, the other aim of this Group Session was to use contexts different than Group Session 6. For example, Group Session 6 also potentially promoted ideas of both Translation and Relativity; however, the relative scale was unknown and the students needed to create it themselves. Whereas, in this session, the relative scale was already known as a cultural convention and the students would only have to use it, rather than create it.

Teacher-Researcher Instructions: **Ask the students to solve the following problems and explain how they solved them. Ask the students to write a number sentence that they think matches the situation.**

1. The temperature was -2° Fahrenheit in the morning. The temperature rose 8° in the afternoon. What was the temperature after the temperature rose?

2. The temperature was -4° Fahrenheit in the morning. The temperature dropped 7° in the afternoon. What was the temperature after the temperature dropped?

3. The warmest recorded temperature of the North Pole is about 5° Celsius. The warmest recorded temperature of the South Pole is about -9° Celsius. Which place has the warmest recorded temperature? And, how much warmer is it?

Group Session 8

One of the goals of Group Session 8 was to promote Translation or Relativity. Group Session 7 ended with students debating and unsure about (a) subtracting a negative number and (b) coordinating how to the find the distance between two locations. The students were still developing their own rules for subtracting negatives, which were sometimes correct and sometimes incorrect. The students were also learning how to coordinate spaces versus tic marks for finding distance. In fact, the students left zero off of the thermometers they created in Group Session 7, because of their struggle with the spaces versus tic marks. The problems in this Group Session were posed to elicit and continue these conversations. The first, fourth, and fifth tasks in Group Session 8 were modified tasks from Tillema (2012), which involved a turtle moving up and down from rocks above water to swimming below water. The second task in this session was developed to promote the idea that distance can be written as subtraction, even when we use negative numbers. The students had developed rules that conflict with writing distance with negative numbers as subtraction. And, the third task was modified from the third task in Group Session 7. However, the context in this Group Session was changed from temperature to elevation. This contextual design move was made to see if similar conversations from Group Session 7 could emerge out of this context. Unlike temperature, this context does not have a scale that is as widely utilized by children as a cultural convention. That is, we may attach a relative scale to it as adults and, certainly, we apply a scale to elevation similarly as we do to temperature. As this task was designed I was unsure how the students would respond and there was consideration that these tasks needed to be brought back to temperature if the student struggled; however, that ended up not being necessary and the students did attach a scale to the elevation context.

An important prompt for the students was, "Please write a number sentence that matches the situation and not just the answer." And, an important prompt considered was, "Can you write a number sentence that matches the situation, but not the answer?" This was a typical question that emerged throughout the Group Sessions. The students often determined their answer first and wrote the number sentences to match their "answer." That is, students have created their own rules, which were sometimes incorrect, and based their number sentences on this rule to produce the answer that they wanted. Besides promoting Translation and Relativity in Group Session 8, a main aim of this Group Session was to facilitate the students in observing the structure of the problem types and supporting the students in viewing subtraction as distance, which the students were not naturally doing.

This Group Session potentially promoted Translation because there was a turtle that was swimming in the water or moving from one rock to another (see, e.g., Tasks 1, 2, & 4). Similarly, this Group Session potentially promoted the relativity conceptual model because there was a scale provided for the students. And, there were problems that involved distance without a directed movement explicit (see, e.g., Task 3 & 5).

Teacher-Researcher Directions: **Ask the students to consider the following situations. Ask the students to solve the following problem and write a number sentence that they think fits the situation.**

1. A helicopter is hovering 8 feet above the surface of the lake. A turtle swims 5 feet below the surface of the lake to hunt. How far apart are the helicopter and the turtle?

2. Consider the following situations. Solve each of the problems. Write a number sentence for each of the situations. Also, look for patterns between each of the five situations.

    a. A helicopter is hovering 10 feet above the surface of the lake. A turtle climbs up on a rock to sunbathe, which is 8 feet above the surface of the lake. How far apart are the helicopter and the turtle?
    b. A helicopter is hovering 10 feet above the surface of the lake. A turtle climbs up on a different rock to sunbathe, which is 6 feet above the surface of the lake. How far apart are the helicopter and the turtle?
    c. A helicopter is hovering 10 feet above the surface of the lake. A turtle climbs up on a different rock to sunbathe, which is 3 feet above the surface of the lake. How far apart are the helicopter and the turtle?
    d. A helicopter is hovering 10 feet above the surface of the lake. A turtle climbs up on a different rock to sunbathe, which is 2 feet above the surface of the lake. How far apart are the helicopter and the turtle?
    e. A helicopter is hovering 10 feet above the surface of the lake. A turtle swims 3 feet below the surface of the lake to hunt. How far apart are the helicopter and the turtle?

3. It is 2° Fahrenheit in Minneapolis, Minnesota. It is -9° Fahrenheit in Chicago, Illinois. What is the difference in temperatures?

4. A turtle is swimming 5 feet beneath the surface of the lake. He spies a fish to eat and dives to 9 feet beneath the surface of the lake. What is the difference between the turtle's two depths?

5. A turtle is hunting for fish 8 feet beneath the surface of the lake. After his hunt, he decides to sun on a rock 12 feet above the surface of a lake. What is the altitude difference between the turtle's two positions?

Group Session 9

The purpose of Group Session 9 was to promote Relativity by providing contexts that used the integers are relative numbers, without movement which was utilized in Group Session 7 and Group Session 8. We had spent some time in the previous two groups session discussing the distance between relative numbers with contexts that incorporated moving between two relative numbers. The purpose of this Group Session was to see how and if students used or applied relative numbers in contexts that do not also promote movement, or Translation. In Group Session 8, it seemed as if students were applying Translation to make sense of Relativity. That is, rather than counting the distance between two relative numbers or writing the two relative numbers a subtraction number sentence to find the distance, the students would re-state the problem as with Translation. For example, instead of using $1 - -3 = 4$ to make sense of a context, the students would reason with $-3 + 4 = 1$ instead. The first two problems of this Group Session came directly from the problems that were part of the Individual Sessions. The types of problems used in this Group Session typically did not go well during the Individual Sessions. On one hand, I thought this might be because Relativity may be the most abstract and the most challenging CMIAS to understand. On the other hand, I thought this might be because these words problems are semantically more challenging to pose. It was important to spend more time on these problems in a group setting to help understand the ways that these student thought about these problems, which are more conceptually and semantically challenging than the other problems posed in the Group Sessions. Three contextual problems were given that utilized games as contexts, being behind in points or being down in runs. The first problem was a context about a two-player game. The scores of the games were not provided, just information about one player's points in relation to the other player. The second problem was similar to the first problem; however, this problem utilized the context of baseball and being down runs. Because the students had previously struggled with these problems, a third problem was created that was different than the first two in that it included a context that involved three players. This problem was designed to draw attention to establishing what zero is by providing a third player that could be treated as "zero." In this case, the player's score below this third player could be treated as a negative integer and the player's score above this third player could be treated as a positive integer. The fourth problem in this Group Session, although it did not promote Relativity, included the students telling stories for open number sentences for two problem types that involved subtraction. Since the students had been struggling with subtraction problems and this was our last Group Session together, this task was used because there was some time left.

***Teacher-Researcher Directions*: Ask the students to consider the following situation. Ask the students to solve the problem and write a number sentence that they think matches it.**

1. Kyle and Justin were playing an online game where points earned are displayed online. Kyle felt competitive and noticed that in the first game he was 5 points behind Justin. In the second game, Kyle lost to Justin by 11 points. How does Kyle start the third game?

    a. The following students wrote numbers sentences for this situation. Which student or students do you agree with and why?

Cara	5 + 11 = □
Julie	-5 – 11 = □
Travis	-5 + -11 = □
Brandon	-5 + 11 = □

Teacher-Researcher Questions:

- *Do you know what the score of the game is?*
- *What does it mean to be down runs and then gain?*
- *How can you use negative numbers? What do the negative numbers mean?*

2. The Red Birds and Dragons were playing against each other in a baseball game. The Red Birds were down 4 runs in the first inning and gained 6 runs against the Dragons in the second inning. How did the Red Birds start the third inning?

3. Gwen, Neal, and Van are playing a game. Gwen has two points less than Neal. Van has 9 points more than Neal. Who is winning? How many more point does the loser need to catch up to the winner?

4. Tell a story for each of the following number sentences:
   a. $2 - 10 = \square$
   b. $-2 - 10 = \square$

APPENDIX E

INDIVIDUAL CONTEXT SESSION 1

Consider the number sentences below. For each number sentence, tell a story that you think matches the number sentence and then solve the number sentence.

1. $5 + 14 = \square$

2. $\square + 12 = 17$

3. $\square - 9 = 22$

4. $29 - 13 = \square$

5. $14 - \square = 5$

6. $\square - 6 = 11$

Consider the following scenarios. Solve the problem and explain your reasoning. Write a number sentence that you think matches the scenario.

7. There were 14 juice boxes in the refrigerator. Julia took 5 juice boxes out of the refrigerator. How many juice boxes are left in the refrigerator?
(take away)

8. Jae has 55 jellybeans. Michael has 67 jellybeans. How many more jellybeans does Michael have from Jae? (compare)

9. Ella decided to go running on a trail. She started running on the trail where there was a signpost that said "Mile 5." She stopped running when she saw a signpost that said "Mile 13." How far did she run?
(distance)

Tell me a story for the following number sentences.

- The researcher should ask the student to pose as many stories as possible for each number sentence.

10. $-17 + 12 = \square$

11. $-2 - 3 = \square$

12. $-5 + \square = -7$

For each of the following problems, write a number sentence that you think matches the situation. Provide a solution to the problem and include any work or pictures that helped you think about the problem.

13. Edward lost 12 pencils in his messy office. While he was cleaning his office he found 5 pencils. How many pencils are still lost in his messy office?
(Bookkeeping)

    - Show the students the following number sentences on cards. Ask if others match.

    - After probing questions for another number sentence show them each of these number sentences, omitting ones that they have provided, and ask: "Last week in other interviews with students, they gave me some other number sentences. Can you tell you tell me what you think about these students were thinking?"

    - Probing questions: Does this number sentence match this question? How does it match? Or, why doesn't it match?

    a. $12 - 5 = \square$
    b. $-12 + 5 = \square$
    c. $12 + -5 = \square$
    d. $5 - 12 = \square$

14. You had 5 sad days and 2 happy days one week. What kind of week did you have?
    (Counterbalance)

    - Show the students the following number sentences on cards. Ask if others match.

    - After probing questions for another number sentence show them each of these number sentences, omitting ones that they have provided, and ask: “Last week in other interviews with students, they gave me some other number sentences. Can you tell you tell me what you think about these students were thinking?”

    - Probing questions: Does this number sentence match this question? How does it match? Or, why doesn’t it match?

    a. $5 + 2 = \square$
    b. $5 – 2 = \square$
    c. $-5 + 2 = \square$
    d. $5 + -2 = \square$
    e. $2 – 5 = \square$

15. It is 2° Fahrenheit outside. The wind chill makes it feel 10° colder. What temperature does it feel like? (Translation)

    - Show the students the following number sentences on cards. Ask if others match.

    - After probing questions for another number sentence show them each of these number sentences, omitting ones that they have provided, and ask: “Last week in other interviews with students, they gave me some other number sentences. Can you tell you tell me what you think about these students were thinking?”

    - Probing questions: Does this number sentence match this question? How does it match? Or, why doesn’t it match?

    a. $10 – 2 = \square$
    b. $-10 + 2 = \square$
    c. $10 + -2 = \square$
    d. $2 – 10 = \square$

16. Kim and Jordan were playing an online game where points earned are displayed online. Kim felt competitive and noticed that in the first game she was 2 points behind Jordan. For the second game, Kim lost to Jordan by 3 points. How does Kim start the third game? (Relativity)
    - Show the students the following number sentences on cards. Ask if others match.

    - After probing questions for another number sentence show them each of these number sentences, omitting ones that they have provided, and ask: "Last week in other interviews with students, they gave me some other number sentences. Can you tell you tell me what you think about these students were thinking?"

    - Probing questions: Does this number sentence match this question? How does it match? Or, why doesn't it match?

a. $2 + 3 = \square$
b. $-2 + 3 = \square$
c. $-3 + 2 = \square$
d. $-2 + -3 = \square$

APPENDIX F

INDIVIDUAL OPEN NUMBER SENTENCE SESSION 1

1. $-20 + 15 = \square$
2. $12 + -16 = \square$
3. $-4 + \square = 10$
4. $7 + \square = -2$
5. $\square + -3 = 7$
6. $\square + 13 = -5$
7. $-8 + -7 = \square$
8. $-2 + \square = -10$
9. $\square + -9 = -16$
10. $10 - 12 = \square$
11. $1 - \square = 3$
12. $-5 - 4 = \square$
13. $2 - -3 = \square$
14. $-1 - \square = 8$
15. $2 - \square = -10$
16. $\square - -1 = 6$
17. $\square - 8 = -5$
18. $-15 - -4 = \square$
19. $-12 - \square = -13$
20. $\square - -2 = 1$

APPENDIX G

INDIVIDUAL CONTEXT SESSION 2

Part 1:

Tell me a story for the following number sentence. As we change the number sentence, tell me how your story changes.

1. Tell me a story for 9 + 6.
    a. How would your story change if we changed the number sentence from 9 + 6 to -9 + 6?
    b. How would that story change if we changed the number sentence from 9 + 6 to -9 + -6?
    c. How would that story change if we changed the number sentence from 9 + 6 to 9 + -6?
    d. How would that story change if we changed the number sentence form 9 + 6 to -9 – -6?

Tell me a story for the following number sentences.

After the student tells a story, ask them if they can tell another story for the same number sentence.

2. -20 + 13 = $\square$

3. -4 – 7 = $\square$

4. -5 + $\square$ = -7

For each of the following problems, write a number sentence for the situation. Provide a solution to the problem.

5. Raquel needs twenty-two cupcakes for her party. She only baked twelve cupcakes. What's Raquel's situation? (Bookkeeping Story)

After the students write a number sentence, show them cards with the following number sentences.

a. $22 - 12 = \square$
b. $-22 + 12 = \square$
c. $22 + -12 = \square$
d. $12 - 22 = \square$
e. $-22 + -12 = \square$

6. Josiah did twenty-two good deeds and eight bad deeds at school this week. What kind of week did Josiah have? (Counterbalance Story)

   After the students write a number sentence, show them cards with the following number sentences.

   a. $22 + 8 = \square$
   b. $22 - 8 = \square$
   c. $22 + - 8 = \square$
   d. $-22 + 8 = \square$
   e. $-22 + -8 = \square$
   f. $8 - 22 = \square$

7. It was 8° Fahrenheit outside. The temperature dropped 20° Fahrenheit. What is the temperature now? (Translation Story)

   After the students write a number sentence, show them cards with the following number sentences.

   a. $20 - 8 = \square$
   b. $-8 + 20 = \square$
   c. $8 - 20 = \square$
   d. $20 + -8 = \square$
   e. $8 + -20 = \square$

8. The Red Birds and Dragons were playing against each other in a baseball game. The Red Birds were down 4 runs in the first inning and gained 6 runs against the Dragons in the second inning. How did the Red Birds start the third inning? (Relativity Story)

After the students write a number sentence, show them cards with the following number sentences.

a. $4 + 6 = \square$
b. $6 – 4 = \square$
c. $-6 + 4 = \square$
d. $-4 + 6 = \square$
e. $-4 + -6 = \square$

9. Angela and her dad decided to plant a tree in their backyard. Angela found a spot in the yard that she wanted to plant the tree. Her dad told her to move the tree from that spot to the right 10 feet. Angela moved the tree to the position that her dad wanted, but then she decided to move it 12 feet to the left. Where did Angela eventually end up planting the tree? (Relativity/Translation Story)

   After the students write a number sentence, show them cards with the following number sentences.

   a. $12 – 10 = \square$
   b. $10 – 12 = \square$
   c. $10 + - 12 = \square$
   d. $-10 + 12 = \square$
   e. $-10 + -12 = \square$
   f. $-12 – 10 = \square$

APPENDIX H

INDIVIDUAL OPEN NUMBER SENTENCE SESSION 2

1. $-16 + 4 = \square$
2. $20 + -33 = \square$
3. $-6 + \square = 15$
4. $-6 + \square = -1$
5. $\square + -2 = 17$
6. $\square + 19 = -4$
7. $-12 + -5 = \square$
8. $-4 + \square = -19$
9. $\square + -9 = -21$
10. $5 - 9 = \square$
11. $4 - \square = 6$
12. $-9 - 8 = \square$
13. $3 - -4 = \square$
14. $-2 - \square = 9$
15. $6 - \square = -10$
16. $\square - -1 = 4$
17. $\square - 9 = -3$
18. $-11 - -2 = \square$
19. $-15 - \square = -16$
20. $\square - -3 = 2$
21. $\square - -4 = 0$
22. $12 + \square = 8$
23. $5 + \square = -3$

APPENDIX I

INDIVIDUAL CONTEXT SESSION 3

Part 1:

Tell me a story for the following number sentence. As we change the number sentence, tell me how your story changes.

1. Tell me a story for 10 + 4.
   a. How would your story change if we changed the number sentence from 10 + 4 to -10 + 4?
   b. How would that story change if we changed the number sentence from 10 + 4 to -10 + -4?
   c. How would that story change if we changed the number sentence from 10 + 4 to 10 + -4?
   d. How would that story change if we changed the number sentence from 10 + 4 to -10 – -4?
   e. How would that story change if we changed the number sentence from 10 + 4 to 10 – -4?

Tell me a story for the following number sentences.

After the student tells a story, ask them if they can tell another story for the same number sentence.

2. $-15 + 4 = \square$

3. $-3 - 5 = \square$

4. $-1 + \square = -2$

For each of the following problems, write a number sentence for the situation. Provide a solution to the problem.

5. Edward lost 12 pencils in his messy office. While he was cleaning his office he found 5 pencils. How many pencils are still lost in his messy office? (Bookkeeping Story)

After the students write a number sentence, show them cards with the following number sentences.

a. $12 - 5 = \square$
b. $-12 + 5 = \square$
c. $12 + -5 = \square$
d. $5 - 12 = \square$
e. $-5 + -12 = \square$

6. Sophie did nine bad deeds and six good deeds at school this week. What kind of week did Sophie have? (Counterbalance Story)

After the students write a number sentence, show them cards with the following number sentences.

a. $9 + 6 = \square$
b. $9 - 6 = \square$
c. $9 + -6 = \square$
d. $-9 + 6 = \square$
e. $-9 + -6 = \square$
f. $6 - 9 = \square$

7. It was 6° Fahrenheit outside. The temperature dropped 11° Fahrenheit. What is the temperature now? (Translation Story)

After the students write a number sentence, show them cards with the following number sentences.

a. $11 - 6 = \square$
b. $-6 + 11 = \square$
c. $6 - 11 = \square$
d. $11 + -6 = \square$
e. $6 + -11 = \square$

8. Emma and Kirsten were playing an online game where points earned are displayed online. Emma felt competitive and noticed that in the first game she was 4 points behind Kirsten. In the second game, Emma lost to Kirsten by 7 points. How does Emma start the third game? (Relativity Story)

   After the students write a number sentence, show them cards with the following number sentences.

   a. $4 + 7 = \square$
   b. $7 - 4 = \square$
   c. $-4 + 7 = \square$
   d. $-7 + 4 = \square$
   e. $-4 + -7 = \square$

9. Angela and her dad decided to plant a tree in their backyard. Angela found a spot in the yard that she wanted to plant the tree. Her dad told her to move the tree from that spot to the right 6 feet. Angela moved the tree to the position that her dad wanted, but then she decided to move it 18 feet to the left. Where did Angela eventually end up planting the tree? (Relativity/Translation Story)

   After the students write a number sentence, show them cards with the following number sentences.

   a. $18 - 6 = \square$
   b. $6 - 18 = \square$
   c. $6 + -18 = \square$
   d. $-6 + 18 = \square$
   e. $-6 + -18 = \square$
   f. $-18 - 6 = \square$

APPENDIX J

INDIVIDUAL OPEN NUMBER SENTENCE SESSION 3

1.	$-18 + 12 = \square$
2.	$15 + -24 = \square$
3.	$-3 + \square = 14$
4.	$-9 + \square = -3$
5.	$\square + -4 = 13$
6.	$\square + 25 = -2$
7.	$-17 + -6 = \square$
8.	$-5 + \square = -21$
9.	$\square + -9 = -17$
10	$12 - 18 = \square$
11	$3 - \square = 4$
12	$-5 - 3 = \square$
13.	$1 - -3 = \square$
14	$-2 - \square = 10$
15	$4 - \square = -12$
16	$\square - -2 = 5$
17	$\square - 6 = -2$
18	$-12 - -4 = \square$
19.	$-10 - \square = -11$
20.	$\square - -3 = 1$
21.	$\square - -5 = 0$
22.	$15 + \square = 9$
23.	$8 + \square = -5$
24.	$\square + 2 = 0$
25.	$-4 - 10 = \square$

APPENDIX K

INDIVIDUAL CONTEXT SESSION 4

Part 1:

Tell me a story for the following number sentences.

After the student tells a story, ask them if they can tell another story for the same number sentence.

1. -12 + 7 = ☐
2. -2 – 10 = ☐
3. -7 + -2 = ☐
4. 1 – -2 = ☐

For each of the following problems, write a number sentence for the situation. Provide a solution to the problem.

5. Jose needs eighteen pizzas for his party. He only bought seven pizzas. What's Jose's situation? (Bookkeeping Story)
6.

   After the students write a number sentence, show them cards with the following number sentences.

   a. 18 – 7 = ☐
   b. -18 + 7 = ☐
   c. 18 + -7 = ☐
   d. 18 – 7 = ☐
   e. -18 + -7 = ☐

7. You had 4 sad days and 3 happy days one week. What kind of week did you have? (Counterbalance Story)

   After the students write a number sentence, show them cards with the following number sentences.

a. $4 + 3 = \square$
b. $4 – 3 = \square$
c. $4 + -3 = \square$
d. $-4 + 3 = \square$
e. $-4 + -3 = \square$
f. $3 – 4 = \square$

8. It was 4° Fahrenheit outside. The temperature dropped 12° Fahrenheit. What is the temperature now? (Translation Story)

   After the students write a number sentence, show them cards with the following number sentences.

   a. $12 – 4 = \square$
   b. $-4 + 12 = \square$
   c. $4 – 12 = \square$
   d. $12 + -4 = \square$
   e. $4 + -12 = \square$

9. The Red Birds and Dragons were playing against each other in a baseball game. The Red Birds were down 10 runs in the first inning and gained 12 runs against the Dragons in the second inning. How did the Red Birds start the third inning? (Relativity Story)

   After the students write a number sentence, show them cards with the following number sentences.

   a. $10 + 12 = \square$
   b. $12– 10 = \square$
   c. $-12 + 10 = \square$
   d. $-10 + 12 = \square$
   e. $-10 + -12 = \square$

10. It is -4° Fahrenheit in Siberia in the morning. It is -7° Fahrenheit in Siberia in the afternoon. What's the difference in temperatures? (Relativity Story)

    After the students write a number sentence, show them cards with the following number sentences.

    a. $7 – 4 = \square$
    b. $4 – 7 = \square$
    c. $-7 – -4 = \square$
    d. $-4 – -7 = \square$
    e. $-7 + 4 = \square$

11. Angela and her dad decided to plant a tree in their backyard. Angela found a spot in the yard that she wanted to plant the tree. Her dad told her to move the tree from that spot to the right 12 feet. Angela moved the tree to the position that her dad wanted, but then she decided to move it 20 feet to the left. Where did Angela eventually end up planting the tree? (Relativity/Translation Story)

After the students write a number sentence, show them cards with the following number sentences.

a. $20 - 12 = \square$
b. $12 - 20 = \square$
c. $12 + -20 = \square$
d. $-12 + 20 = \square$
e. $-12 + -20 = \square$
f. $-20 - 12 = \square$

APPENDIX L

INDIVIDUAL OPEN NUMBER SENTENCE SESSION 4

1.	$-20 + 15 = \square$
2.	$12 + -16 = \square$
3.	$-4 + \square = 10$
4.	$-7 + \square = -2$
5.	$\square + -3 = 7$
6.	$\square + 13 = -5$
7.	$-8 + -7 = \square$
8.	$-2 + \square = -10$
9.	$\square + -9 = -16$
10	$10 - 12 = \square$
11	$1 - \square = 3$
12	$-5 - 4 = \square$
13.	$2 - -3 = \square$
14	$-1 - \square = 8$
15.	$2 - \square = -10$
16	$\square - -1 = 6$
17	$\square - 8 = -5$
18	$-15 - -4 = \square$
19.	$-12 - \square = -13$
20.	$\square - -2 = 1$
21.	$\square - -3 = 0$
22.	$17 + \square = 8$
23.	$6 + \square = -2$
24.	$\square + 4 = 0$
25.	$-2 - 8 = \square$

APPENDIX M

EXCERPT OF TEACHER-RESEARCHER JOURNAL

A – Individual Open Number Sentence Session 2

It seemed that for the first time Abby was able to solve problems like 2 – 5. Although, problems that Abby could solve correctly before she was now solving incorrectly. For example -20 + 15 she was saying was -35, when in her pre-assessment she was saying -5. At the end of Group Session #3 there seemed to be come confusion from problems that supported Counterbalance. This could have created this misconception. Although A can solve problems like -11 – -2 easily and efficiently, the strategies for these types of problems appear to have caused misconceptions for problems of the type -10 – 2.

The way that A uses the words "bigger" and "smaller" are not how adults use bigger and smaller. J and K also use bigger and smaller like A. Here's an example:

$$-12 + -5 = \square$$

A: (Writes 12 + 5 vertically. Then writes -17 in the box.)

T: Can you explain what you're thinking?

A: A negative plus a negative (points at -12 and -5) equals a negative (points at -17). So I just took the negatives off and I did twelve plus five and I got seventeen.

T: Ok. How do you know that a negative plus a negative is a negative?

A: Because it wouldn't make sense if (points at -12) there was two negatives and it equaled a whole number.

T: How come it wouldn't make sense?

A: Because if you are adding, if you are adding onto to a negative it gets bigger. And, so it wouldn't get smaller and go into a whole number.

A gets a subtracting a negative correct for 4 – $\square$ = 6; but does not get subtracting a negative correct for 3 – -4 = $\square$. The drawing she provides is not a number line, but a number path. Her explanation is not clear, however, as I tried to probe further.

$$4 - \square = 6$$

A: (Draws four tally marks/lines. Thinks for a bit and draws two more tally marks lower and to the right. Then writes -2 in the box.) I did four minus negative two and I got six because ... I did the four right here (points at the upper tally's) and the two (points at the lower tally's). And, then this is six.

T: Ok. Can you should me where the four is in your problem?

A: (Points at the upper tally's.)

T: There. Ok. And, where's the six at?

A: (Uses pen and circles all of the tally's.)

T: Ok. And then, where's this minus a negative two coming?

A: (Points at the lower tally's.)

T: Right there. Ok. So, how'd you know it was negative two that's in the box?

A: Well, because I did two ... If I did it backwards (moves pen across 4 - -2). If I did two plus four and I got six. So then I thought that it would be negative two.

T: So can you explain what you mean by backwards? If you did it backwards ...

A: If like six (points at 6) minus two would give your four. So I thought four minus two would give you six.

T: Ok.

It seems that the role of subtracting is not clear. When subtracting a negative from a negative, -a – -b when a > b, the students can use quantity, can conceptualize subtraction as take-away, and even draw on analogies with positive numbers. However, this reasoning does not extend to –a – b. And, the takeaway model breaks down. Students need to think about distance (relativity) or translations to make sense of –a – b. For example, A's reasoning became a typical struggle:

$$-9 - 8 = \square$$

A: (Writes -1 in the box.)

T: Ok. Can you explain that?

A: I did nine minus eight and I got negative one.

T: How'd you do that?

A: Because negative (draws tally marks)...There's nine, nine, and then eight (crosses off eight tally's). And then there's one (circles one tally mark) negative left over.

T: Ok.

Drawings, especially drawings with tally marks or lines, is how A reasons about addition and subtraction of integers. Interestingly, sometimes these are just quantities and the tallies do not have order. Sometimes, however, A will give the tallies order and values. They are a "number line hybrid" at times, but not all of the time.

There is robust and logical thinking grounded in the rules that the students are making up for themselves. In the following example, A reflects on the strategy of making analogies to whole numbers and makes a connection about the magnitude of the numbers.

$$-11 - -2 = \square$$

A: (Draws tally's. Writes -9 in the box.)

T: How'd you think of that?

A: Well, if it's two negatives it equals a negative, so I just took the negative off. And, I did eleven minus two and I got nine.

T: Ok. How come you just take the negatives off?

A: Because there's two negatives. They're both negative. And, I know if... Since two isn't bigger than eleven that it's not going to go into a whole number.

The use of words "up/down" and "plus/minus" is interesting and I think should be looked at throughout all of the interviews. Two examples of "up/down" and "plus/minus" from A are shown below.

Example 1:

$$-15 - \square = -16$$

A: (Writes 1 in the box.) Well, fifteen plus ... well if you do it fifteen (points at -15) plus (points at minus sign) one (points at 1), negative fifteen minus one is negative sixteen because it's add up if it a whole number because it was a negative (points at 1) it wouldn't add. But, it adds up and fifteen (points at -15) plus one (points at 1) is sixteen (points at -16). And, also fifteen, negative fifteen (starts writing a vertical problem) plus or minus (writes minus) would be sixteen.

T: Ok. So you were saying things like it adds up to negative sixteen. Can you explain that to me?

A: Because one (points at 1, it's not a negative. And if it's not a negative then it goes up, like it, or it goes down. It doesn't go up.

T: Ok. What do you mean it goes down and doesn't go up? What are you talking about?

A: Like it goes down to sixteen (points at -16). It doesn't go up to fourteen.

T: Ok.

Example 2:

$$-6 + \square = -1$$

T: Can you solve it for me and tell me what you are thinking?

A: (Writes -5 in the box.)

T: Ok.

A: Well, negative, five plus six ... Well, actually it'd just be five and not negative five. Well, five ... if there's (starts drawing) six and then if you are adding, you are going up to whole numbers. So since it's negative so cross off five and then there's one left.

T: Ok. So what goes in here (points at box)? Five or negative five?

A: Five.

T: Ok. And so when you said go up to whole numbers, what do you mean?

A: Like go up to one or like two.

T: So like what in this problem tells you that you are going up?

A: Because this number gets smaller (points at -1).

T: Which number gets smaller?

A: (Points at -1) The one.

T: From what?

A: From six (points at -6).

T: Ok.

K – Individual Open Number Sentence Session 2

K is not as verbally expressive as the other two participants and she does not draw a lot of pictures. Most of K's computations appear to be mental and it can be a struggle sometimes to get her to share how she thought about things. However, it was very interesting how she was starting to think about negative integers. She appears to heavily draw on translation. For example, for $-16 + 4 = \square$ she references that the four "is not enough" to bring the negative sixteen into the whole numbers. K provides an example of "using fingers" to compute negative integers computations.

$$-6 + \square = 15$$

K: I started off with negative six. Then I was like negative five (thumb), negative four (next finger), negative three (another finger), negative two (another finger), negative one (another finger), zero (thumb on other hand). Then I counted five: one, two, three, four, five (holds up only four more fingers). Wait, one, two, three, four, yeah. Five (holds up thumb on left hand again). That was eleven. And then since I was at five and I needed to get to fifteen, I added ten more onto eleven and I got twenty-one.

Here is an example of one of K's interesting strategies:

$$\square + -2 = 17$$

K: (Writes 19.) It was a small negative number and it was a big whole number. And, so what I did was two plus seventeen and I got nineteen. And then another way to help me out, to figure out that this was nineteen (points at nineteen), to double check on that, was like negative ... negative two plus two, which is basically like two minus two. I'd have zero. And if you subtract nineteen minus two, you'd have seventeen. And with that I had seventeen left over when I got to zero. You just... yeah.

The students' use of words like "bigger" and "higher" are not the same as what teachers mean often mean by bigger. K refers to the magnitude of the numbers with "bigger" and "higher" in the following examples:

$$\square + 19 = -4$$

K: Hmmm. (Writes on paper.) Nineteen is a big number versus negative four. So nineteen is the number trying to get to the whole numbers, like zero and up. So I knew I needed a bigger number to make sure that nineteen didn't reach the whole number and only get to negative four. So I did nineteen plus four and got twenty-three. And, to double check on that I did minus nineteen and got four.

$$\square + -9 = -21$$

K: (Writes -12 in the box.) Also, this is very similar to the problem right before it. This is a negative number (points at -9) and the total amount, like the answer, is a lot higher than nine. So you taking another negative number here (points at box) to add on to negative nine (points at -9) to get negative twenty-one (points at -21). And I did twenty-one minus nine and got twelve and added a negative sign in front of it.

I asked some probing questions about "bigger."

T: You said something about this (points at -21) being bigger than this (points at -9). Can you explain that? And that's why you chose to add? What do you mean by that?

K: Originally twenty-one in actual numbers is a lot bigger than nine.

T: Hmmm-mmm.

K: And, just nine itself plus zero doesn't equal twenty-one, so I had to take another number to help out the nine.

T: What do you mean in original numbers it's bigger? What do you mean by that?

K: Like (shrugs) like one, two, and up, all the way. Just like the ones we use in real-life of math.

T: Ok. Ok. So is negative twenty-one bigger than negative nine?

K: Yes.

T: Yes. Ok.

K used her fingers to solve the problems sometimes.

$$5 - 9 = \square$$

K: (Quickly writes 4 in the box.)

T: Ok. Can you explain that?

K: Wait. (Whispers) Nine minus five how did I screw it up? (Then just adds a negative sign to the four.) That would be. (Uses fingers and then nods at me.)

T: Ok. So ... can you explain what you were doing with your fingers for me?

K: I figured out that five was in the first position (points at 5).

T: Mmm-hmm.

K: The next number, which was nine, was bigger. So, then I knew it was going to be a negative number. So I just used my fingers and go like (with left hand uses thumb first, then each finger) four, three, two, one, zero, negative one, negative two, negative three, negative four.

T: Ok.

K: I had one left over and I knew I used nine fingers.

When it comes to solving open number sentences, the students start to develop rules that involve "adding" when it is a subtraction problem (e.g., -9 – 8) or "subtracting" when it's an addition problem (e.g., -12 + 4). Sometimes they do this productively, see K's example:

T: mmm-hmm. Then how did you know to an addition problem because I don't see a plus sign there?

K: I think that when it involves a negative number and then a minus in a subtraction problem, that's like adding more onto the negative number. So yeah...

It appeared to me that she was trying to make an analogy/comparison of subtracting -2 – -11 to 2 - 11. It appeared that her reasoning was, I know that 2 – 11 is -9. So, -2 – -11 would have to be 9.

K: Originally, if you just did negative two plus eleven that would equal nine because also if you did eleven minus nine you'd get two. So that's how you can double check that.

T: Yeah, but that says negative eleven? Can you help me understand that?

K: It's not an addition problem. Like I said, a negative number minus another negative number is like adding. So like if you did, two minus eleven that time... it's like a reverse with subtraction.

T: So what's two minus eleven?

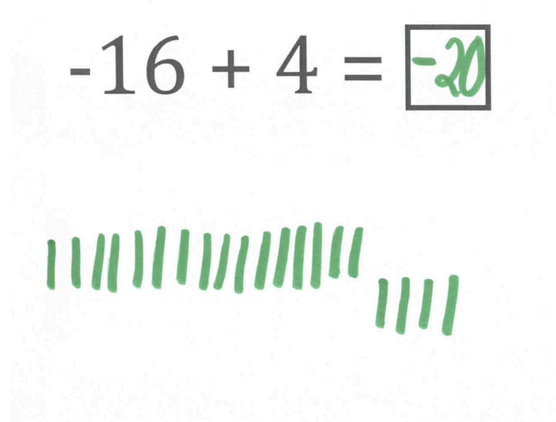
K: Two minus eleven would be negative nine.

T: Ok.

K: I think. And then if you reverse it, negative two minus negative eleven you'd get nine. It's

APPENDIX N

EXAMPLE UNIT OF DATA

Transcript (Word Use)	Drawings (Visual Mediator)
A: (Draws sixteen "tally marks" or lines. Then, draws four more to the right of the original sixteen, but lower. Then writes -20 in the box.) T: Ok. Can you tell me what you are thinking? A: Well, I did sixteen lines. And, then I added four. And then sixteen plus four would be twenty and it's negative twenty. T: How come it's negative twenty? A: Because I added onto to negative. T: Ok. So what do these represent (drags fingers along 16 tally marks)? A: Sixteen. T: Sixteen. And what do these represent (drags fingers along the four lower tallies)? A: Four. T: How come you did them separate like that? A: So I knew that it was two different numbers. T: Ok. What do you mean by two different numbers? A: That they are, like once you add them, that they're one. But they weren't like the ...It wasn't just twenty, it sixteen and four. T: Ok.	

APPENDIX O

SAMPLE PHASE 2 CODING SHEET

Individual Open Number Sentence Session 1		Researcher 1	Researcher 2	Agreed Upon	Visual Mediators	Comments
1.	-20 + 15 = □	Translation	Translation	Translation	Answers crossed off and the one in the box only	Comparison to degrees
2.	12 + -16 = □	Translation	Translation/Rule	Translation/Rule	Answer in the box only	Comparison to degrees & 16 - 12
3.	-4 + □ = 10	Translation	Translation	Translation	Answer in the box only	
4.	-7 + □ = -2	Rule	Rule	Rule	Answer in the box only	
5.	□ + -3 = 7	Translation	Translation	Translation	Answer in the box only	
6.	□ + 13 = -5	Rule	Translation/Rule	Translation/Rule	Answer in the box only	Refers to getting from one number to another with a rule
7.	-8 + -7 = □	Rule	Rule	Rule	Answer in the box only	Analogy to 8 - 7 (doesn't work)
8.	-2 + □ = -10	Rule	Rule	Rule	Answer in the box only	Analogy to 2 + 8 (does work)
9.	□ + -9 = -16	Rule	Rule	Rule	Answer in the box only	Analogy to 7 + 9 (does work)
10	10 – 12 = □	Rule	Rule	Rule	Answer in the box only	Can't do 10 - 12 like 12 - 10
11	1 – □ = 3	No Answer	Rule	Rule	No solution	Implicit Rule: You can't substract and get a bigger answer
12	-5 – 4 = □	Rule	Rule	Rule	Answer in the box only	Analogy to 5 - 4 (doesn't work)
13.	2 – -3 = □	Translation	Translation	Translation	Answer in the box only	two to three is one, but it's negative (doesn't work)
14	-1 – □ = 8	Translation	Translation	Translation	Answer in the box only	Getting from -1 to 8 (doesn't work because ignores minus sign)
15.	2 – □ = -10	Rule	Rule	Rule	Answer in the box only	Analogy to 8 + 2 (doesn't work)
16	□ – -1 = 6	Translation	Translation/Rule	Translation	Answer in the box only	Compares to -1 - ? = 8
17	□ – 8 = -5	Rule	Rule	Rule	Answer in the box only	Analogy to 13 - 8 = 5 (doesn't work)
18	-15 – -4 = □	Rule	Rule	Rule	Answer in the box only	Analogy to 15 - 4 (does work)
19.	-12 – □ = -13	No Answer	No Answer	N/A	Answer in the box only	Can't verbalize
20.	□ – -2 = 1	Rule	Rule	Rule	-1 then the negative is crossed off to become 1	Had the right answer, but then changes
		Agreed	16			
		Disagreed	4			
		Percentage of Agreement	0.8			

APPENDIX P

SAMPLE PHASE 4 CODING SHEET

Individual Open Number Sentence Session 1		**Researcher 1**	**Researcher 2**	**Agreed Upon**	**B**	**C**	**T**	**A**	**AR**	**P**
1.	-20 + 15 = □	Translation	Translation				1			
2.	12 + -16 = □	Analogy/ Translation	Analogy/ Translation				1	1		
3.	-4 + □ = 10	Translation	Translation				1			
4.	-7 + □ = -2	Analogy	Analogy					1		
5.	□ + -3 = 7	Translation/ Analogy	Translation/ Analogy				1	1		
6.	□ + 13 = -5	Translation	Translation				1			
7.	-8 + -7 = □	Analogy/ Translation	Translation/ Analogy				1	1		
8.	-2 + □ = -10	Analogy	Analogy					1		
9.	□ + -9 = -16	Analogy	Analogy/ Algebraic Reasoning	Analogy				1		
10	10 – 12 = □	Analogy	Analogy/ ?					1		
11	1 – □ = 3	Counterbalance?	Analogy/ ?	Analogy				1		
12	-5 – 4 = □	Analogy	Analogy					1		
13.	2 – -3 = □	Translation	Translation				1			
14	-1 – □ = 8	Translation	Translation				1			
15.	2 – □ = -10	Analogy	Translation/ Anlaogy	Analogy				1		
16	□ – -1 = 6	Analogy/Translation	Analogy/ Translation				1	1		
17	□ – 8 = -5	Analogy	Analogy					1		
18	-15 – -4 = □	Proceduralizaiton/Analogy	Analogy/ Proceduralization?					1		1
19.	-12 – □ = -13	N/A	N/A							
20.	□ – -2 = 1	N/A	Proceduralization	N/A						
				Counts	0	0	9	13	0	1
				Percentages	0%	0%	45%	65%	0%	5%